THE QUEST FOR FUNCTIONS

Paul Vaderlind, Stockholm University

January, 2005

Functional Equations for The Beginners

INTRODUCTION

Functional equations is a rather popular topic at the IMO and other mathematical competitions, both national and international. At least 19 IMO-problems can be classified as functional equations and all these problems are listed below. The question posed in this type of problems is to find all functions satysfying the given equation and, possibly, some additional conditions like continuity, monotonicity or being bounded.

There however is no general method of solving this kind of problems and the present text offers only some basic ideas that may turn out to be useful. Some type of tricks are used when the functions considered are $f : \mathbb{N} \to \mathbb{N}$, other for funktions $f : \mathbb{Q} \to \mathbb{Q}$, and still different methods for $f : \mathbb{R} \to \mathbb{R}$. Yet another approach may be used when we know that the functions are looking for are polynomials. (Throughout this text $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denotes the sets of positive integers, integers, rational numbers and real numbers respectively. An additional + sign, like $\mathbb{R}^+$, means "positive". $\mathbb{N}_0$ denotes the set of all non-negative integers $\{0, 1, 2, 3, ...\}$.)

As an exemple of a functional equation, consider the famous Cauchy's equation $f(x + y) = f(x) + f(y)$, where $f : \mathbb{R} \to \mathbb{R}$ and $x, y \in \mathbb{R}$. This very general equation has in fact a very limmited family of solutions as soon as one add som extra constrain. For example, if one demands that the solution has to be a continuous function then the only solutions are the *trivial* ones: the linear functions $f(x) = cx$, for any real constant c

Even if one demands the continuty only in one sigle point $x_0 \in \mathbb{R}$, or if one asks for f bounded in some interval $(a, b) \subset \mathbb{R}$, or monotone, then the equation has still only the trivial solutions. In order to find some non-trivial solutions one has to look beyond Lebesgue measurable functions and that such pathological solutions exist was proved by G. Hamel (in Math. Ann. 60, (1905), 459-462).

The list of all functional equation that occurred at the IMO is the following (many more such problems has made to the IMO short-lists):

1968.5. Let f be a real-valued function defined for all real numbers, such that for some $a > 0$ we have

$$f(x + a) = \frac{1}{2} + \sqrt{f(x) - f(x)^2} \text{ for all } x.$$

Prove that f is periodic, and give an example of such a non-constant f for $a = 1$.

1972.5. $f(x)$ and $g(x)$ are real-valued functions defined on the real line. For all x and y, $f(x+y)+f(x-y) = 2f(x)g(y)$, f is not identically zero and $|f(x)| \leq 1$ for all x. Prove that $|g(x)| \leq 1$ for all x.

1975.6. Find all polynomials $P(x, y)$ in two variables such that:

(1) $P(tx, ty) = t^n P(x, y)$ for some positive integer n and all real t, x, y:

(2) for all real $x, y, z : P(y+z, x) + P(z+x, y) + P(x+y, z) = 0$;
(3) $P(1, 0) = 1$.

1977.6. The function $f(x)$ is defined on the set of positive integers and its values are positive integers. Given that $f(n+1) > f(f(n))$ for all n, prove that $f(n) = n$ for all n.

1981.6. The function $f(x, y)$ satisfies:
$f(0, y) = y+1, f(x+1, 0) = f(x, 1)$ and $f(x+1, y+1) = f(x, f(x+1, y))$ for all non-negative integers x, y. Find $f(4, 1981)$.

1982.1. The function $f(n)$ is defined on the positive integers $\mathbb{N}$ and takes non-negative integer values. Moreover $f(2) = 0, f(3) > 0, f(9999) = 3333$ and for all $m, n \in \mathbb{N}$: $f(m+n) - f(m) - f(n) = 0$ or 1. Determine $f(1982)$.

1983.1. Find all functions f defined on the set of positive real numbers $\mathbb{R}^+$ which take positive real values and satisfy:
$f(xf(y)) = yf(x)$ for all x, y; and $f(x) \to 0$ as $x \to \infty$.

1986.5. Find all functions f defined on the non-negative real numbers and taking non-negative real values such that: $f(2) = 0, f(x) \neq 0$ for $0 \leq x < 2$, and $f(xf(y))f(y) = f(x+y)$ for all x, y.

1987.4. Prove that there is no function f from the set of non-negative integers $\mathbb{N}_0$ into itself such that $f(f(n)) = n + 1987$ for all $n \in \mathbb{N}_0$.

1988.3. A function f is defined on the positive integers $\mathbb{N}$ by:
$f(1) = 1, f(3) = 3, f(2n) = f(n), f(4n+1) = 2f(2n+1) - f(n)$, and
$f(4n+3) = 3f(2n+1) - 2f(n)$ for all $n \in \mathbb{N}$.
Determine the number of positive integers $n \leq 1988$ for which $f(n) = n$.

1990.4. Construct a function from the set of positive rational numbers into itself such that $f(xf(y)) = \dfrac{f(x)}{y}$ for all x, y.

1992.2. Find all functions f defined on the set of all real numbers with real values, such that $f(x^2 + f(y)) = y + f(x)^2$ for all x, y.

1993.5. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(1) = 2, f(f(n)) = f(n) + n$ for all $n \in \mathbb{N}$, and $f(n) < f(n+1)$ for all $n \in \mathbb{N}$?

1994.5. Let S be the set of all real numbers greater than -1. Find all functions $f\colon S \to S$ such that $f\bigl(x + f(y) + xf(y)\bigr) = y + f(x) + yf(x)$ for all x, y, and $\dfrac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

1996.3. Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f\bigl(m+f(n)\bigr) = f\bigl(f(m)\bigr)+f(n)$ for all $m, n \in \mathbb{N}_0$.

1998.6. Consider all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(t^2 f(s)) = sf(t)^2$ for all $s, t \in \mathbb{N}$. Determine the least possible value of $f(1998)$.

1999.6. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$ for all x, y in $\mathbb{R}$.

2002.5. Find all real-valued functions on the set of real numbers $\mathbb{R}$ such that
$(f(x) + f(y))((f(u) + f(v)) = f(xu - yv) + f(xv + yu)$
for all $x, y, u, v \in \mathbb{R}$.

2004.2. Find all polynomials $P(x)$ with real coefficients which satisfy the equality
$P(a - b) + P(b - c) + P(c - a) = 2P(a + b + c)$
for all real numbers a, b, c such that $ab + bc + ca = 0$.

Most of these problems are considerate in this text. Otherwise, the complete solutions may be found on the Web, at http://www.kalva.demon.co.uk/imo.html. However I suggest that the reader try to solve the problems on his own, before consulting the proposed solutions.

SOME EASY TRICKS

1. Transformation of variables.

This is a really basic trick and may be used as a part of a solution of a more complex problem. Generally, given an equation of a type $f(g(x)) = h(x)$, with $g(x), h(x)$ given functions, then, if $g(x)$ has an inverse then, letting $t = g(x)$, we get $f(x) = h(g^{-1}(x))$.

Let's solve the following equation:

Example 1. Find all functions $f(x)$ defined for all real numbers, such that $f\left(\frac{x+1}{x}\right) = 1 + \frac{1}{x} + \frac{1}{x^2}$ for all $x \neq 0$.

Solution. By letting $t = \frac{x+1}{x}$, we get $x = \frac{1}{t-1}$. Hence, after some calculations, the equation reduces to $f(t) = t^2 - t + 1$. Thus $f(x) = x^2 - x + 1$. $\square$

2. Creating simultaneous equations.

This is another simple trick, which often works when the equation involves two values $f(g(x))$ and $f(h(x))$, for two different algebraic expressions $g(x)$ and $h(x)$. Consider the equation:

Example 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $x^2 f(x) + f(1-x) = 2x - x^4$ for all $x \in \mathbb{R}$.

Solution. Replacing x by $1-x$, we have $(1-x)^2 f(1-x) + f(x) = 2(1-x) - (1-x)^4$. Since $f(1-x) = 2x - x^4 - x^2 f(x)$ by the given equation, substituting this into the last equation and solving for $f(x)$, we get $f(x) = 1 - x^2$.

Now we should check that this function satisfiy the given equation: $x^2 f(x) + f(1-x) = x^2(1-x^2) + \big(1 - (1-x)^2\big) = 2x - x^4$. $\square$

One more example:

Example 3. Solve the equation $f\left(\frac{1}{x}\right) + \frac{1}{x} f(-x) = x$, where f is a real valued function defined for all real numbers except 0.

Solution. Replacing x by $\frac{1}{x}$ yelds $f(x) + xf\left(-\frac{1}{x}\right) = \frac{1}{x}$. Replacing now x by $-x$ leads to a new equation $f(-x) - xf\left(\frac{1}{x}\right) = -\frac{1}{x}$. From this equation and the original one can we now find the function $f(x)$: Multiply the first equation with x and add to the last one. What we get is $2f(-x) = x^2 - \frac{1}{x}$. Replacing once again x by $-x$ we get the final answer: $f(x) = \frac{x^3+1}{2x}$. It remains to verify that this function satisfies the given eqation. $\square$

Remark. In most cases we solve the equation under the (silent) assumption that the function $f(x)$ *exists*. As a consequence, it is necessary to check that the obtai-ned function really satisfies the given equation.

3. Using symmetry.

If possible, one should use symmetry when dealing with the equation involving more than one variable.

Example 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x+y) = x + f(y)$ for all $x, y \in \mathbb{R}$.

Solution. Left-hand side of the equation is symmetric in x and y. Thus $x + f(y) = f(x+y) = f(y+x) = y + f(x)$, which can be written as $f(x) - x = f(y) - y$, for all $x, y \in \mathbb{R}$. Hence $f(x) - x$ is constant for all $x \in \mathbb{R}$, and the answer is $f(x) = x + c$, for any choice of the real constant c, provided that those functions satisfy the given equation. This however can be easily checked. $\square$

Example 5. Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x+y) - f(x-y) = 4xy$ for all $x, y \in \mathbb{R}$.

Solution. Let $u = x + y$ and $v = x - y$. Then the equation can be written as $f(u) - f(v) = u^2 - v^2$, or $f(u) - u^2 = f(v) - v^2$. Since this relation holds for arbitrary $u, v \in \mathbb{R}$ then $f(u) - u^2$ is constant. Thus, $f(u) - u^2 = c$ and the answer is the family of functions $f(x) = x^2 + c$, for any choice of a real constant c, provided these functions satisfy the given equation. This however is easy (although necessary) to check. $\square$

4. Evaluating $f(x_0)$ for some special choices of x_0.

Finding $f(x_0)$ for some values of x_0, like $f(0)$, $f(1)$, $f(2)$, $f(-1)$ and so on, may give some ideas on the structure of $f(x)$. This seems to be specially usefull when the equation involves more than one variable.

Example 6. (Korea, 1988) Find $f : \mathbb{R} \to \mathbb{R}$, such that $f(x)f(y) = f(xy) + x + y$ for all $x, y \in \mathbb{R}$.

Solution. Letting $y = 0$ we get $f(x)f(0) = f(0) + x$. Hence, $f(0) \neq 0$ and $f(x) = \dfrac{x}{f(0)} + 1$. Taking now $x = 0$ we find that $f(0) = 1$. Thus $f(x) = x + 1$ and it is easy to verify that this function satisfies the given eqation. $\square$

In the next example, let $\mathbb{Q}^+$ denote the set of positive rational numbers.

Example 7. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$, such that $f\left(x + \dfrac{y}{x}\right) = f(x) + \dfrac{f(y)}{f(x)} + 2y$, for all $x, y \in \mathbb{Q}^+$.

Solution. By letting (x, y) be $(1,1), (1,2)$ and $(2,2)$ we find out that $f(2) = f(1) + 3$, $f(3) = f(1) + \dfrac{f(2)}{f(1)} + 4$ and $f(3) = f(2) + 5$. From these three equalities we can deduce that $f(1) = 1, f(2) = 4$ and $f(3) = 9$. This leads to the hypothesis that $f(n)=n^2$, for at least $n\in\mathbb{N}$.

This hypothesis may be now verified by taking $x=y=n$ and using the obtained relation $f(n+1)=f(n)+1+2n$ together with the mathematical induction.

We may now suspect that the only solution of the equation is the function $f(x)=x^2$, for all $x\in\mathbb{Q}^+$. Let's take first $x=n$, $y=m$ and then $x=\dfrac{m}{n}$, $y=m$ (for $n,m\in\mathbb{N}$). We get

$$f\Big(n+\frac{m}{n}\Big)=f(n)+\frac{f(m)}{f(n)}+2m=n^2+\frac{m^2}{n^2}+2m \text{ and}$$

$$f\Big(\frac{m}{n}+n\Big)=f\Big(\frac{m}{n}\Big)+\frac{f(m)}{f\big(\frac{m}{n}\big)}+2m=f\Big(\frac{m}{n}\Big)+\frac{m^2}{f\big(\frac{m}{n}\big)}+2m.$$

From the last two equalities it follows that $n^2+\dfrac{m^2}{n^2}=f\Big(\dfrac{m}{n}\Big)+\dfrac{m^2}{f\big(\frac{m}{n}\big)}$, which can be expressed as

$$0=f\big(\frac{m}{n}\big)-\frac{m^2}{n^2}-n^2+\frac{m^2}{f\big(\frac{m}{n}\big)}=f\big(\frac{m}{n}\big)-\frac{m^2}{n^2}-\frac{n^2}{f\big(\frac{m}{n}\big)}\Big(f\big(\frac{m}{n}\big)-\frac{m^2}{n^2}\Big)=\Big(f\big(\frac{m}{n}\big)-\frac{m^2}{n^2}\Big)\Big(1-\frac{n^2}{f\big(\frac{m}{n}\big)}\Big).$$

Let $\dfrac{p}{g}\in\mathbb{Q}^+$, where $p,q\in\mathbb{N}$. If $1-\dfrac{q^2}{f\big(\frac{p}{q}\big)}\neq 0$ then, according to the equality above, $f\big(\dfrac{p}{q}\big)-\dfrac{p^2}{q^2}=0$, i.e. $f\big(\dfrac{p}{q}\big)=\dfrac{p^2}{q^2}=\big(\dfrac{p}{q}\big)^2$.

If $1-\dfrac{q^2}{f\big(\frac{p}{q}\big)}=0$ then, $\dfrac{f(2q)}{f\big(\frac{2p}{2q}\big)}=\dfrac{4q^2}{f\big(\frac{p}{q}\big)}\neq\dfrac{q^2}{f\big(\frac{p}{q}\big)}=1$. Thus $\dfrac{f(2q)}{f\big(\frac{2p}{2q}\big)}\neq 1$, and then, letting $n=2q$, $m=2p$ into the equality above, we find again that $f\big(\dfrac{p}{q}\big)=f\big(\dfrac{2p}{2q}\big)=\dfrac{(2p)^2}{(2q)^2}=\big(\dfrac{p}{q}\big)^2$.

Hence the answer is $f(x)=x^2$, and it is easy to verify that this function satisfies the equation. $\square$

5. Polynomials.

When the functions we are looking for are polynomials there are several several properties one should take into the consideration. The most importatnt are: the degree, the finite number of zeroes (unless the polynomial is the trivial one: $p(x) \equiv 0$) and the Factor Theorem (stating that $p(\alpha) = 0$ if and only if $x - \alpha$ is a divisor of $p(x)$).

Example 8. Find all real polynomials $p(x)$ such that $p(x+1)+2p(x-1) = 6x^2+5$ for all $x \in \mathbb{R}$.

Solution. First we observe that $p(x)$ has to be of degree 2, hence we may write $p(x) = ax^2 + bx + c$. Substituting this expresion into the equation we get $a(x+1)^2 + b(x+1) + c + 2a(x-1)^2 + 2b(x-1) + 2c = 6x^2 + 5$, which reduces to $3ax^2 + (-2a+3b)x + (3a-b+3c) = 6x^2 + 5$. Identifying the coefficients gives $a = 2,\ b = \frac{4}{3},\ c = \frac{1}{9}$. Hence $p(x) = 2x^2 + \frac{4}{3}x + \frac{1}{9}$ and verification that this polynomial satisfies the given relation is an easy task. $\square$

Example 9. Find all real polynomials $p(x)$ such that $xp(x-1) = (x-2)p(x)$ for all $x \in \mathbb{R}$.

Solution. Letting $x = 0$ we get $0 = -2p(0)$, i.e. $p(0) = 0$. Similarily, for $x = 2$ we get $p(1) = 0$. Hence $p(x)$ is divisible by x and by $(x-1)$ and we can write $p(x) = x(x-1)q(x)$, where $q(x)$ is a polynomial of degree 2 less than the degree of $p(x)$.

Replacing $p(x)$ with $x(x-1)q(x)$ in the original equation gives $x(x-1)(x-2)q(x-1) = (x-2)x(x-1)q(x)$ for all $x \in \mathbb{R}$. Hence $q(x-1) = q(x)$ for all $x \in \mathbb{R}$.

Let now x_0 be any fixed real number and consider the polynomial $h(x) = q(x) - q(x_0)$. It is obvious that $h(x_0) = 0$. Moreover, $h(x_0+1) = q(x_0+1) - q(x_0) = q(x_0) - q(x_0) = 0$, and, usig the induction, one can show that $h(x_0+n) = 0$ for all $n \in \mathbb{Z}$.

Since a non-zero polynomial only has a finite number of zeroes, then $h(x) \equiv 0$. It implies that $q(x)$ is a constant polynomial, and then $p(x) = cx(x-1)$, for any choice of a real constant c.

It only remains to check that the polynomials $p(x) = cx(x-1)$ satisfy the original equation. $\square$

MORE SOFISTICATED METHODS

8. Continous functions.

Some equations involving continous functions $f : \mathbb{R} \to \mathbb{R}$ may be solved in the following way: Find first some special values, like $f(0)$ or $f(1)$. By induction determine then the values $f(n)$ for all $n \in \mathbb{N}$, followed by the values $f(n)$ for all $n \in \mathbb{Z}$. In the next step find the values $f\big(\dfrac{1}{n}\big)$ for $n \in \mathbb{Z}$ and then find $f\big(\dfrac{m}{n}\big)$ for all $\dfrac{m}{n} \in \mathbb{Q}$. Finally, use the *continuity* of $f(x)$ and the fact that the set of rational numbers is *dense* in $\mathbb{R}$, to determaine the formula for $f(x)$ for all $x \in \mathbb{R}$.

That the the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$ means that for each $x \in \mathbb{R}$ there exists a sequence $\{x_n\}$ of rational numbers such that $\lim\limits_{n\to\infty} x_n = x$.

Suppose a function $f(x)$ is defined on the subset $I \subset \mathbb{R}$ Then we say that $f(x)$ is continuous at a point $x_0 \in I$ if, for each sequence $\{x_n\} \subset I$ such that $\lim\limits_{n\to\infty} x_n = x_0$, we have $\lim\limits_{n\to\infty} f(x_n) = f(\lim\limits_{n\to\infty} x_n) = f(x_0)$.

We say that $f(x)$ is continous on I if it is continous at each point $x_0 \in I$.

As an illustration consider the already mentioned continous version of Cauchy's equation:

Example 10. Find all continous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

Solution. Letting $x = y = 0$ into the equation we get $f(0) = 0$. By induction one shows easily that $f(nx) = nf(x)$, for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$. Hence $f(n) = nf(1)$, for all $n \in \mathbb{N}$.

If we in the equation let $y = -x$ then we get $f(0) = f(x) + f(-x)$. Thus $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. For $n \in \mathbb{N}$ we have then $f(-n) = -f(n) = -nf(1)$, which means that $f(n) = nf(1)$ is valid for all $n \in \mathbb{Z}$.

Suppose now that $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. Then $n \cdot f\big(\dfrac{m}{n}\big) = f\big(n \cdot \dfrac{m}{n}\big) = f(m) = mf(1)$. Thus $f\big(\dfrac{m}{n}\big) = \dfrac{m}{n} f(1)$, i.e. $f(x) = xf(1)$ is valid for all $x \in \mathbb{Q}$.

Suppose finally that $x \in \mathbb{R}$ but $x \notin \mathbb{Q}$. Then, since $\mathbb{Q}$ is dense in $\mathbb{R}$, consider a sequence $\{x_n\}$ od rational numbers such that $\lim_{n\to\infty} x_n = x$. By the continuity of $f(x)$ we have $f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \big(x_n f(1)\big) = f(1) \lim_{n\to\infty} (x_n) = f(1) \cdot x$.

Hence $f(x) = xf(1)$ for all $x \in \mathbb{R}$. Since $f(1)$ can be any real number then the solution, if exists, must be of the form $f(x) = cx$ for any real constant c. As usual, it remains to verify that these functions satisfy the original equation. $\square$

Example 11. Find all continous functions $f(x)$ defined for $x > 0$ and such that $f(x+y) = \dfrac{f(x)f(y)}{f(x)f(y)}$ for all $x, y \in \mathbb{R}^+$.

Solution. It is obvious that $f(x) \neq 0$ for all $x \in \mathbb{R}^+$. Taking $x = y$ we get $f(2x) = \dfrac{f(x)f(x)}{f(x)+f(x)} = \dfrac{1}{2}f(x)$. For $y = 2x$ we have then $f(3x) = \dfrac{f(x)f(2x)}{f(x)+f(2x)} = \dfrac{f(x)\frac{1}{2}f(x)}{f(x)+\frac{1}{2}f(x)} = \dfrac{1}{3}f(x)$. This suggest that $f(nx) = \dfrac{1}{n}f(x)$ for all $n \in \mathbb{N}$, and may easily be shown by induction.

By taking $x = 1$ in the last equality we get $f(n) = \dfrac{1}{n}f(1)$ for all $n \in \mathbb{N}$. Moreover, $f(1) = f\big(n \cdot \frac{1}{n}\big) = \frac{1}{n}f\big(\frac{1}{n}\big)$, which means that $f\big(\frac{1}{n}\big) = nf(1)$. Then, for all $m, n \in \mathbb{N}$, $f\big(\frac{m}{n}\big) = f\big(m \cdot \frac{1}{n}\big) = \frac{1}{m}f\big(\frac{1}{n}\big) = \frac{n}{m}f(1)$.

We have that far shown that $f(x) = \dfrac{1}{x}f(1)$ for all $x \in \mathbb{Q}^+$. Using the continuity argument this can be extended to all $x \in \mathbb{R}^+$. The answer is then $f(x) = \dfrac{c}{x}$ for every non-zero real constant c. $\square$

Example 12. (Croatia, 1996) Suppose t is a fixed number such that $0 < t < 1$. Find all functions $f : \mathbb{R} \to \mathbb{R}$, continous at $x = 0$, such that $f(x) - 2f(tx) + f(t^2x) = x^2$ for all $x \in \mathbb{R}$.

Solution. Since the equation can be written as $\big(f(x)-f(tx)\big)-\big(f(tx)-f(t^2x)\big) = x^2$, we may start by a substitution $g(x) = f(x) - f(tx)$. This will simplify the equation to $g(x) - g(tx) = x^2$.

Now, since $f(x)$ is contious at $x = 0$, then is obvious that even $g(x)$ is continous at $x = 0$ and that $g(0) = f(0) - f(0) = 0$.

In the equation $g(x) - g(tx) = x^2$ we now sibstitute x by tx several times, getting successively:

$$g(x) - g(tx) = x^2,$$
$$g(tx) - g(t^2x) = t^2x^2,$$
$$g(t^2x) - g(t^3x) = t^4x^2,$$
....
$$g(t^{n-1}x) - g(t^nx) = t^{2(n-1)}x^2.$$

Adding all thos equalities we find that $g(x) - g(t^nx) = \big(1 + t^2 + t^4 + ... + t^{2(n-1)}\big)x^2$, and, since $t^2 \neq 1$ then $g(x) - g(t^nx) = x^2\dfrac{1 - t^{2n}}{1 - t^2} = x^2\dfrac{1}{1 - t^2} - x^2\dfrac{t^{2n}}{1 - t^2}$.

Remembering that $0 < t < 1$ we can now let $n \to \infty$. Then $t^nx \to 0$ as well as $\dfrac{t^{2n}}{1 - t^2} \to 0$ and, using the continuity of $g(x)$ at $x = 0$, we get $g(x) - g(0) = x^2\dfrac{1}{1 - t^2}$. Since $g(0) = 0$ then finally $g(x) = \dfrac{x^2}{1 - t^2}$.

We have that far found out that $f(x) - f(tx) = \dfrac{x^2}{1 - t^2}$ for all $x \in \mathbb{R}$. What we can do now is to repeat the same procedure we did above: substitution x by tx several times. We get:

$$f(x) - f(tx) = \frac{x^2}{1 - t^2},$$
$$f(tx) - f(t^2x) = \frac{t^2x^2}{1 - t^2},$$
$$f(t^2x) - f(t^3x) = \frac{t^4x^2}{1 - t^2},$$
....
$$g(t^{n-1}x) - g(t^nx) = \frac{t^{2(n-1)}x^2}{1 - t^2}.$$

Adding those equations we find out that $f(x) - f(t^nx) = \dfrac{x^2}{1 - t^2}\big(1 + t^2 + t^4 + ... + t^{2(n-1)}\big) = \dfrac{x^2}{1 - t^2} \cdot \dfrac{1 - t^{2n}}{1 - t^2}$.

Letting now $n \to \infty$ and using the continuity of $f(x)$ at $x = 0$, we get $f(x) - f(0) = \dfrac{x^2}{(1 - t^2)^2}$ for all $x \in \mathbb{R}$.

Thus the only possible solutions are the functions $f(x) = \dfrac{x^2}{(1-t^2)^2} + c$ for any choice of a real constant c. Now one must just check that those functions really satisfy the given equation, which in fact turn out to be the case. $\square$.

9. Additional insights.

The methods described above are unfortunately not sufficient for solving more difficult problems of the IMO type. Some additional knowledge about the functions we are looking for is necessary and the question one should ask could be the following:

a) Is the function even? Is it odd? (In those cases it will be sufficient to consider only $x > 0$.)

b) Is the function periodic? (if "yes", then it is sufficient to limit the domain of the function to some finite interval.)

c) Is the function one-to-one (injective)? Is it onto (surjective)?

d) Does there exist any fixed point (i.e. such x that $f(x) = x$)?

e) Is there any symmetry?

f) When dealing with functions defined on $\mathbb{N}$ then the uniquness of decomposition into prime factors may turn out to be useful.

g) It's good to be aware of the alternative representation of non-negative integers in bases other than 10. The binary representation (in base 2) is quite useful (see for example the problem 3 from the IMO 1988).

h) Again, it is sometimes useful to be aware that any non-empty subset of $\mathbb{N}$ has the least element.

Applications of some of these ideas are illustrated in the following examples.

Example 13. Consider all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+4) + f(x-4) = f(x)$ for all $x \in \mathbb{R}$. Show that any such function is periodic and that there is a least common positive period p for all of them. Find p.

Solution. Putting $x + 4$ instead of x we get $f(x+8) + f(x) = f(x+4)$. Adding this equaition to the original one reduces to $f(x+8)+f(x-4) = 0$. Putting again

$x+4$ instead of x yelds $f(x+12)+f(x)=0$. Since then $f(x+24)+f(x+12)=f\big((x+12)+12\big)+f(x+12)=0$ the the last two equations impliy that $f(x+24)=f(x)$ for all $x\in\mathbb{R}$.

Thus we have found a common period $p=24$ for all $f(x)$ satisfying the original equation. Now the question is if this period is the least positive one.

Consider the function $f(x)=\sin\dfrac{\pi x}{12}$. Since 2π is the least positive period of $\sin x$ then $p=24$ is the least positive period of $f(x)$. At the same time it is easy to show that $f(x)$ satisfies the condition of the problem: $\sin\dfrac{\pi(x+4)}{12}+\sin\dfrac{\pi(x-4)}{12}=\sin\left(\dfrac{\pi x}{12}+\dfrac{\pi}{3}\right)+\sin\left(\dfrac{\pi x}{12}-\dfrac{\pi}{3}\right)$, which, by easy trigonometry, reduces to $\sin\dfrac{\pi x}{12}$.

Hencet the least common period is $p=24$. □

Example 14. (Romania, 1999) Suppose that the function $f:\mathbb{N}\to\mathbb{N}$ is surjective, while the function $g:\mathbb{N}\to\mathbb{N}$ is injective. Given that $f(n)\geq g(n)$ for all $n\in\mathbb{N}$, prove that $f=g$.

Solution. Let $A=\{n\in\mathbb{N}:\ f(n)\neq g(n)\}$ and suppose A is a non-empty subset of $\mathbb{N}$. Then the set $B=\{g(n)\ :\ n\in A\}$ is also a non-empty subset of $\mathbb{N}$ and thus has the least element. Suppose $g(a)$, for some $a\in A$, is the least element of b. Then, since $g(n)$ is injective, we have $g(a)<g(b)$, for all $a\neq b\in A$ and, by the definition of A, $g(a)<f(a)$.

Since $f(n)$ is surjective then there exists $c\in\mathbb{N}$ such that $f(c)=g(a)<f(a)$. Note that $c\neq a$. Now, since $g(n)$ is injective then $g(c)\neq g(a)=f(c)$. Hence $c\in A$ and we have $g(c)<f(c)=g(a)$, which contradicts the choice of a. Thus, the set A is empty, which means thet $f(n)=g(n)$ for all $n\in\mathbb{N}$, □

Example 15. (IMO, 1983) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that
$f(xf(y))=yf(x)$ for all $x,y\in\mathbb{R}^+$, and $f(x)\to 0$ as $x\to\infty$.

Solution. By taking $x=y=1$ we get $f(f(1))=f(1)$. Taking $x=1$ and $y=f(1)$ yelds $f(f(f(1)))=\big(f(1)\big)^2$. Combining these two equalities we receive $\big(f(1)\big)^2=f(f(f(1)))=f(f(1))=f(1)$. Hence $f(1)\big(f(1)-1\big)=0$. Since $f(1)>0$, then we must have $f(1)=1$, i.e. $x=1$ is a fixed point of $f(x)$.

Taking $y = x$ yelds $f(xf(x)) = xf(x)$, which means that $xf(x)$ are fixed points of f for all $x \in \mathbb{R}^+$.

Suppose that $f(x)$ has a fixed point $x_0 > 1$. Then, by the above, $x_0 f(x_0) = x_0^2$ is a fixed point as well. Then again, $x_0^2 f(x_0^2) = x_0^4$ is a fixed point of f, and, by induction, x_0^{2k} are fixed points of f for all $k \in \mathbb{N}$. Since $x_0 > 1$ then $\lim\limits_{k\to\infty} x_o^{2k} = \infty$ and it follows that $\lim\limits_{k\to\infty} f\left(x_o^{2k}\right) = \lim\limits_{k\to\infty} x_o^{2k} = \infty$, which contradicts the condition stated in the problem. Thus $f(x)$ has no fixed points greater than 1.

Let's now check if $f(x)$ has some fixed points within the interval $(0, 1)$. If x_0 is such a point then, taking $y = x_0$ and $x = \dfrac{1}{x_0}$ into the relation we get $1 = f(1) = f\big(\dfrac{1}{x_0} \cdot x_0\big) = f\big(\dfrac{1}{x_0} f(x_0)\big) = x_0 f\big(\dfrac{1}{x_0}\big)$, i.e. $f\big(\dfrac{1}{x_0}\big) = \dfrac{1}{x_0}$. Thus $\dfrac{1}{x_0} > 1$ is a fixed point of $f(x)$, which contradicts the previous result. Hence $x = 1$ is the only fixed point of $f(x)$.

We have however fund earlier that $xf(x)$ are fixed points of f for all $x \in \mathbb{R}^+$. Thus $xf(x) = 1$ for all $x \in \mathbb{R}^+$, which means that $f(x) = \dfrac{1}{x}$. It is now easy to check that this function satisy the given conditions. $\square$

Example 16. (IMO, 1987). Prove that there is no function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(f(n)) = n + 1987$ for all $n \in \mathbb{N}_0$.

Solution. Suppose there is such a function $f(x)$. Then $f(x)$ must be injective (one-to-one) because $f(a) = f(b)$ would imply $a = f(f(a)) - 1987 = f(f(b)) - 1987 = b$. Moreover, it is clear that the function $f(f(n)) = n + 1987$ will never have the values from the set $\{0, 1, 2, 3, ..., 1986\}$, and those 1987 numbers are the only one from $\mathbb{N}_0$ that the function $f(f(n))$ will miss ($\star$).

Suppose now that $f(n)$ misses exactly k distinct values $c_1, c_2, ..., c_k$ in $\mathbb{N}_0$, i.e. $f(n) \neq c_1, c_2, ..., c_k$ for all $n \in \mathbb{N}_0$. This implies that $f(f(x))$ misses the following $2k$ values: $c_1, c_2, ..., c_k, f(c_1), f(c_2), ..., f(c_k)$ in $\mathbb{N}_0$. (Note that all the numbers $f(c_j)$ are distinct, since f is injective.)

Now, if $w \notin \{c_1, c_2, ..., c_k, f(c_1), f(c_2), ..., f(c_k)\}$, then there is $m \in \mathbb{N}_0$ such that $f(m) = w$. Since $w \neq f(c_1), f(c_2), ..., f(c_k)$ and $m \neq c_1, c_2, ..., c_k$ so there is $n \in \mathbb{N}_0$ such that $f(n) = m$. Hence $f(f(n)) = w$.

This proves that the function $f(f(n))$ misses only the $2k$ values $\{c_1, c_2, ..., c_k, f(c_1), f(c_2), ..., f(c_k)\}$ and no others. This contradicts the fact stated as ($\star$) above

(1987 is an odd number). $\square$

Example 17. (IMO, 1968). Let $f(x)$ be a real-valued function defined for all real numbers x, such that for some positive constant a the equation

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - \big(f(x)\big)^2} \text{ holds for all } x \in \mathbb{R}.$$

Prove that $f(x)$ is periodic, and, for $a = 1$, give an example of such a non-constant function $f(x)$.

Solution. One way of solving this problem is to rewrite the equation as $f(x + a) - \frac{1}{2} = \sqrt{f(x)\big(1 - f(x)\big)}$, and to realize that both sides of the equality are "symmetrical" about $\frac{1}{2}$. Then it seems natural to make the substitution $g(x) = f(x) - \frac{1}{2}$.

With this substitution we will have $g(x) \geq 0$ and $\Big(g(x+a)\Big)^2 = \frac{1}{4} - \Big(g(x)\Big)^2$ for all x. It follows that that $\Big(g(x+2a)\Big)^2 = \frac{1}{4} - \Big(g(x+a)\Big)^2 = \frac{1}{4} - \Big(\frac{1}{4} - \big(g(x)\big)^2\Big) = \Big(g(x)\Big)^2$. Thus $g(x+2a) = g(x)$ for all x.

Hence, $f(x+2a) = g(x+2a) + \frac{1}{2} = g(x) + \frac{1}{2} = f(x)$ so $f(x)$ is periodic with the period $2a$.

There are several examples of non-constant functions satisfying the given equation and having period $= 2$. One such eaxample is $f(x) = \frac{1}{2}\Big(1 + \big|\cos\frac{\pi x}{2}\big|\Big)$ (check this!!). An another example one can get by taking $f(x)$ to be arbitrary in the interval $[0,1)$ (for example, let $f(x) = 1$ for $0 \leq x < 1$), then let $f(x) = \frac{1}{2}$ for $1 \leq x < 2$. Finally use the equality $f(x+2) = f(x)$ to extend $f(x)$ to all other values of $x \in \mathbb{R}$. $\square$

Example 18. (IMO, 1996). Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f\big(m + f(n)\big) = f\big(f(m)\big) + f(n) \text{ for all } m, n \in \mathbb{N}_0.$$

Solution. Taking $m = n = 0$, we get $f\big(f(0)\big) = f\big(f(0)\big) + f(0)$, which implies that $f(0) = 0$. Taking $m = 0$. we get $f\big(f(n)\big) = f(n)$, i.e $f(n)$ is a fixed point of $f(x)$ for all $n \in \mathbb{N}$. As a consequence, the equation becomes $f\big(m+f(n)\big) = f(m)+f(n)$ $(\star)$.

Now we will show by induction that if n_0 is a fixed point of $f(x)$ then kn_0 is also a fixed point of $f(x)$ for all $k \in \mathbb{N}_0$. We know this already for $k=0$ and $k=1$. If we assume that kn_0 is a fixed point of $f(x)$ for some $k \in \mathbb{N}_0$ then $f\big((k+1)n_0\big) = f\big(kn_0+n_0\big) = f\big(kn_0+f(n_0)\big) = f(kn_0)+f(n_0) = kn_0+n_0 = (k+1)n_0$, and so is $(k+1)n_0$ also a fixed point of $f(x)$.

If 0 is the only fixed point of $f(x)$ then, by the relation $(\star)$, $f(m)=0$ for all $m \in \mathbb{N}_0$.

Otherwise $f(x)$ has a least fixed point $n_0 \neq 0$ (the least element in the set of all non-zero fixed points of $f(x)$). We want to show now that kn_0 are the only fixed points of $f(x)$ (for $k \in \mathbb{N}_0$).

So suppose that x is a fixed point. Then $x \geq n_0$ and dividing x ny n_0 we get $x = kn_0+r$, where $0 \leq r < n_0$. Thus $x = f(x) = f(r+kn_0) = f\big(r+f(kn_0)\big) = f(r)+f(kn_0) = f(r)+kn_0$. From this it follows that $f(r) = x-kn_0 = r$. This means that r is a fixed point of $f(x)$ and by the minimality of n_0, it follows that $r=0$. Hence $x = kn_0$ and we are done.

We have however shown that $f(n)$ are fixed points of $f(x)$ for all $n \in \mathbb{N}_0$. Hence $f(n) = c_n n_0$ for some numbers $c_n \in \mathbb{N}_0$. However $c_0 = 0$ since $0 = f(0) = c_0 n_0$.

Dividing now each $n \in \mathbb{N}_0$ by n_0 we get $n = kn_0 + r$, where $0 \leq r < n_0$. Then $f(n) = f(r+kn_0) = f\big(r+f(kn_0)\big) = f(r)+f(kn_0) = f(r)+kn_0 = c_r n_0 + kn_0 = (c_r+k)n_0 = \Big(c_r + \lfloor \frac{n}{n_o} \rfloor\Big)$, where $\lfloor x \rfloor$ denotes the integer part of x. Hence the answer is $f(n) = \Big(c_r + \lfloor \frac{n}{n_o} \rfloor\Big)$, but this, of course, must be veryfied.

To this end, for each $n_0 > 0$ let $c_0 = 0$ and let $c_1, c_2, ..., c_{n_0-1} \in \mathbb{N}_0$ be arbitrary. The function $f(n) = \Big(c_r + \lfloor \frac{n}{n_o} \rfloor\Big)$, where r is the remainder of n divided by n_0, are all solutions: Write $m = kn_0 + r$ and $n = ln_0 + s$, with $0 \leq r, s < n_0$. Then $f\big(m+f(n)\big) = f\big(r+kn_0+(c_s+l)n_0\big) = c_r n_0 + kn_0 + c_s n_0 + ln_0 = f\big(f(m)\big) + f(n)$. That $f(n) \equiv 0$ also is a solution is obvious. $\square$

10. A good guess.

Sometimes a good guess may simplify the work considerably. There are equations which give a hint what the solution should look like. What then remains is to prove that the guessed solution $f_0(x)$ is unique. One can for example make a substitution $f(x) = f_0(x) + g(x)$, and show thereafter that $g(x) \equiv 0$, or one can even use another methods.

Example 19. Find all polynomials $p(x)$ such that $p(x+1) = p(x) + 2x + 1$ for all $x \in \mathbb{R}$.

Solution. It is easy to guess that $p(x) = x^2$ is a solution to the equation. In order to find if there are other solutions, let $f(x) = p(x) - x^2$. Then the equation translates to $f(x+1) = p(x+1) - (x+1)^2 = p(x) - x^2 = f(x)$ for all $x \in \mathbb{R}$.

By the same method as in Example 8 above we may then show that the only polynomials $f(x)$ satisfying the equation $f(x+1) = f(x)$ for all $x \in \mathbb{R}$ are constant polynomials, $f(x) = c$. Hence the answer to the given equation are all polynomials of the form $p(x) = x^2 + c$ for any choice of the real constant c. However, it is again necessary to check that those polynomials satisfy the given equation. $\square$

Example 20. (Poland, 1992) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that following conditions are satisfied:

(1) $f(-x) = -f(x)$ for all $x \in \mathbb{R}$,
(2) $f(x+1) = f(x) + 1$ for all $x \in \mathbb{R}$ and
(3) $f\left(\frac{1}{x}\right) = \frac{1}{x^2} f(x)$ for all $x \in \mathbb{R}$, $x \neq 0$.

Solution. It is immediate to see that $f(x) = x$ satisfies all the conditions of the problem. But is it the only solution?

Let $g(x) = f(x) - x$. Using (1), (2) and (3) it is easy to find the following properties of $g(x)$:

(4) $g(-x) = -g(x)$,
(5) $g(x+1) = g(x)$ and
(6) $g\left(\frac{1}{x}\right) = \frac{1}{x^2} g(x)$.

From (4) and (5) we find straightforward that $g(0) = g(-1) = 0$. Suppose now that $x \neq 0$ and $x \neq -1$. We find that (the number above the sign of equality indicates which property is being used):

$$g(x) \overset{(5)}{=} g(x+1) \overset{(6)}{=} (x+1)^2 \cdot g\big(\frac{1}{x+1}\big) \overset{(4)}{=} -(x+1)^2 \cdot g\big(\frac{-1}{x+1}\big) \overset{(5)}{=} -(x+1)^2 \cdot g\big(\frac{-1}{x+1}+1\big) = -(x+1)^2 \cdot g\big(\frac{x}{x+1}\big) \overset{(6)}{=} -(x+1)^2 \cdot \frac{x^2}{(x+1)^2} \cdot g\big(\frac{x+1}{x}\big) = -x^2 \cdot g\big(1+\frac{1}{x}\big) \overset{(5)}{=} -x^2 \cdot g\big(\frac{1}{x}\big) \overset{(6)}{=} -x^2 \cdot \frac{1}{x^2} \cdot g(x) = -g(x).$$

Hence $2g(x) = 0$, i.e. $g(x) = 0$ for all $x \in \mathbb{R}$, and $f(x) = x$ is the only solution to the equation. □

Example 21. Show that there are infinitely man functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(2) = 2$ and $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

Solution. Each $n > 1$ has a unique representation as a product of prime numbers, $n = p_1^{n_1} p_2^{n_2} ... p_k^{n_k}$, where p_i are prime numbers and $n_i \in \mathbb{N}$. The condition of the problem implies then that $f(n) = \big(f(p_1)\big)^{n_1}\big(f(p_2)\big)^{n_2}...\big(f(p_k)\big)^{n_k}$ ($\star$). Hence the function is defined by it's values on the set of prime numbers, which may be then choosen arbitrarily.

To exhibit one specific infinite family of solutions let $P = \{q_1, q_2, q_3, ...\}$ be the set of all prime numbers greater that 2, in increasing order. For each $m \in \mathbb{N}$, let the function f_m be defined on P in the following way: $f_m(q_i) = q_{i+m}$. Then we may add $f_m(1) = 1$ and $f_m(2) = 2$, and, using the property ($\star$) extend the definition of f_m to the whole $\mathbb{N}$. □

11. Some useful facts.

We have already worked out one of the Cauchy's equations but there are another three. Since all they are already a folklore, the complet solutions are not given here, only the final answer, and they may be used in the solutions of other problems as given facts. However one should recommend that the reader try to solve those equations on his own.

Note that we only mention the continous solutions. The general cases of Cauchy's equations (i.e. without any extra conditions) are much harder to solve.

The only continuous solutions to the following Cauchy's equations:

(1) $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$,

(2) $f(xy) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^+$,

(3) $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}^+$

are the following families of functions:

(1) $f(x) = c^x$ for any real constant $c > 0$, or $f(x) \equiv 0$,

(2) $f(x) = c \ln x$ for any real constant c,

(3) $f(x) = x^c$ for any real constant c, or $f(x) \equiv 0$.

Example 22. (Example 10 revisited). Find all continous functions $f(x)$ defined for $x > 0$ and such that $f(x+y) = \dfrac{f(x)f(y)}{f(x)f(y)}$ for all $x, y \in \mathbb{R}^+$.

Solution. We note that $f(x) \neq 0$ for all $x \in \mathbb{R}^+$ and put $g(x) = \dfrac{1}{f(x)}$. The the equation may be written as $g(x+y) = g(x) + g(y)$. This is the well known Cauchy's equation and the continous solutions are $g(x) = cx$ for any choice of real constant c. Hence the solutions of the original equation are $f(x) = \dfrac{1}{cx}$ for all non-zero real constants c. $\square$

Example 23. Find all continous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f\big(x^y\big) = f(x)^{f(y)}$ for all $x, y \in \mathbb{R}^+$.

Solution. We note that the constant function $f_0(x) \equiv 1$ is a solution to the equation. Suppose then that there is another solution, $f(x)$, and that $f(a) \neq 1$ for some $a \in \mathbb{R}^+$. Then, for all $x, y \in \mathbb{R}^+$

$$f(a)^{f(xy)} = f\big(a^{xy}\big) = f\Big(\big(a^x\big)^y\Big) = f\big(a^x\big)^{f(y)} = f(a)^{f(x)f(y)},$$

from which follows that $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}^+$. Since this is one of the Cauchy's equations (equation (3) above), we have $f(x) = x^c$ for some real constant $c \neq 0$.

Putting this function into the original equation it is easy to find that $c = 1$.

Hence the equation has two solutions: $f(x) \equiv 1$ and $f(x) = x$. $\square$

Example 24. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation $f(x+y) = f(x) + f(y) + f(x)f(y)$ for all $x, y \in \mathbb{R}$.

Solution. Since the right-hand side of the equation can be written as $f(x)+f(y)+f(x)f(y) = (f(x)+1)(f(y)+1)-1$ then it seems natural to make the substitution $g(x) = f(x) + 1$.

This leads to the (Cauchy's) equation $g(x + y) = g(x)g(y)$ which only has continous solutions of the form $g(x) = c^x$ for any choice of a real constant $c > 0$, or the zero function $g(x) \equiv 0$.

Thus the solutions to the original equation are $f(x) = c^x - 1$ for any choice of a real constant $c > 0$, or the constant function $f(x) = -1$. $\square$

RELATED QUESTIONS

In slightly different, although closely related type of problems we are asked for a specific value $f(a)$ of the function rather than finding the explicit formula for $f(x)$. The function in question is given in a similar form as in problems above. The solving methods are more or less the same as those for solving functional equations.

Example 25. (Hong Kong, 1996) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(1) \neq 0$ and $f(x + y^2) = f(x) + 2\big(f(y)\big)^2$ for all $x, y \in \mathbb{R}$. Find the value of $f(1996)$.

Solution. By taking $x = y = 0$ we find that $f(0) = 0$. Taking $x = 0$ and $y = 1$ yelds $f(1) = f(0) + 2\big(f(1)\big)^2$, so $f(1) = \dfrac{1}{2}$.

Since $f(2) = f(1 + 1^2) = f(1) + 2\big(f(1)\big)^2$ and $f(3) = f(2 + 1^2) = f(2) + 2\big(f(1)\big)^2 = f(1) + 4\big(f(1)\big)^2$, we can guess that $f(n + 1) = f(1) + 2(n - 1)\big(f(1)\big)^2 = \dfrac{1}{2} + 2(n-1)\big(\dfrac{1}{2}\big)^2 = \dfrac{n}{2}$ for all $n \in \mathbb{N}$. This can be easily verified by induction.

Hence $f(1996) = 998$. $\square$

Example 26. (Greece, 1997) Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a function satisfying following conditions:

(1) $f(x)$ is strictly increasing,

(2) $f(x) > -\dfrac{1}{x}$ and

(3) $f(x)f\big(f(x) + \dfrac{1}{x}\big) = 1$ for all $x \in \mathbb{R}^+$.

Find $f(1)$.

Solution. Let $f(1) = a$. Setting $x = 1$ in (3) we get $af(a+1) = 1$. Thus $a \neq 0$ and $f(a+1) = \dfrac{1}{a}$.

Taking now $x = a + 1$ in (3) yelds $\dfrac{1}{a}f\big(\dfrac{1}{a} + \dfrac{1}{a+1}\big) = 1$, which implies that $f\big(\dfrac{1}{a} + \dfrac{1}{a+1}\big) = a = f(1)$. Since $f(x)$ is strictly increasing, we must have $\dfrac{1}{a} + \dfrac{1}{a+1} = 1$. By solving this equation we get $a = \dfrac{1 \pm \sqrt{5}}{2}$.

Suppose that $a = \dfrac{1+\sqrt{5}}{2}$. Then $1 < a = f(1) < f(a+1) = \dfrac{1}{a} < 1$. This contracition implies that $f(1) = a = \dfrac{1-\sqrt{5}}{2}$.

(One may note that a function with described condition really exists, for example $f(x) = \dfrac{1-\sqrt{5}}{2x}$.) $\square$

COLLECTION OF PROBLEMS

The problems below are the first set of problems for training in solving functional equations. To each problem there is given a hint, but it is not necessary to follow it in order to find the solution. There, as almost always, are many different ways to approach a mathematical problem. The suggested complet solutions are given in the next section.

1. Find all solutions $f(x)$ of the equation $xf(x) + 2xf(-x) = -1$ where $x \in \mathbb{R}$ and $x \neq 0$.

(Hint: Create an additional equation.)

2. Find all functions $f(x)$ soving the equation $f(x) + f\Big(\dfrac{1}{1-x}\Big) = x$, where $x \neq 0$ and $x \neq 1$.

(Hint: Create an additional equation.)

3. Solve the functional equation $2f(\tan x) + f(-\tan x) = \sin 2x$, where $f(x)$ are definded in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(Hint: Transformation of variable.)

4. (Poland, 1989) Determine all functions $f : \mathbb{R} \to \mathbb{R}$, such that for all $x, y \in \mathbb{R}$, $(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$.

(Hint: Similar to Example 4.)

5. Find all polynomials $p(x)$ satisfying the relation $p(x+1) = p(x) + 2x + 1$.

(Hint: Discover symmetry.)

6. (Sweden, 1995) Find all polynomials $p(x)$ which solve the following equation for all $x \in \mathbb{R}$: $xp(x-1) = (x-26)p(x)$.

(Hint: The same method as in Example 8.)

7. Determine all continous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 2$ and $f(xy) = f(x)f(y) - f(x+y) + 1$ for all $x, y \in \mathbb{R}$.

(Hint: Find first the expression for $f(x)$ for $x \in \mathbb{Q}$.)

8. (Canada, 2002) Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $xf(y) + yf(x) = (x+y)f(x^2 + y^2)$ for all $x, y \in \mathbb{N}_0$.

(Hint: Try some values of x and guess the solution. Then prove the correctness of your guess.)

9. (Asian-Pacific MO, 2002) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ has only a finite number of roots and $f(x^4 + y) = x^3 f(x) + f\big(f(y)\big)$ for all $x, y \in \mathbb{R}$.

(Hint: Show first that $f(x^4) = x^3 f(x)$ for all $x, y \in \mathbb{R}$. Prove then that $f(x)$ is an odd function. What are the zeros of $f(x)$?)

10. (UK, 1977) Let $f : \mathbb{N} \to \mathbb{N}_0$ satisfy

(a) $f(mn) = f(m) + f(n)$, for all $m, n \in \mathbb{N}$,
(b) $f(n) = 0$ whenever the units digit of n (in base 10) is a '3', and
(c) f(10)=0.

Prove that $f(n) = 0$ for all $n \in \mathbb{N}$.

(Hint: Factorization.)

11. Find all functions $f(x, y)$ from the set $\mathbb{Q}^+ \times \mathbb{Q}^+$ of all *pairs* of positive rational numbers (x, y) to the set $\mathbb{Q}^+$, which satisfy the following conditions:

(1) $f(x, 1) = x$ for all $x \in \mathbb{Q}^+$,
(2) $f(x, x) = 1$ for all $x \in \mathbb{Q}^+$ and
(3) $f(x, y) \cdot f(z, t) = f(xz, yt)$ for all $x, y, z, t \in \mathbb{Q}^+$.

(Hint: No need for that. It's a very easy problem.)

12. Determine all continous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f^2(x) = f(x + y)f(x - y)$ for all $x, y \in \mathbb{R}$.

(Hint: To get rid of the square, take logarithms on both sides.)

13. Find all functions $f : \mathbb{N} \to \mathbb{N}$ which satisfy the equation $f\Big(f\big(f(n)\big)\Big) + f\big(f(n)\big) + f(n) = 3n$, for all $n \in \mathbb{N}$.

(Hint: Show first that $f(n)$ must be injective. What is $f(1)$?)

14. Find all real polynomials $p(x)$ satisfying $p(x^2) + p(x)p(x + 1) = 0$ for all $x \in \mathbb{R}$.

(Hint: Show that if x_0 is a zero of the polynomial $p(x)$ then even x_0^2 is a zero of this polynomial.)

15. (Sweden, 1962) Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ and all $r \in \mathbb{Q}$ the inequality $|f(x) - f(r)| \leq 7(x - r)^2$ is satisfied.

(Hint: Find first the values od $f(x)$ for rational x. The triangle-inequality may be useful.)

16. (Israel, 1995) Let a be a real number. Determine all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that $ax^2f\big(\frac{1}{x}\big)+f(x)=\dfrac{x}{x+1}$ for all $x\in\mathbb{R}^+$.

(Hint: Put $\dfrac{1}{x}$ instead of x and symplify the equation. Consider several cases depending on a.)

17. (Korea, 1999) Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f\big(\dfrac{x-3}{x+1}\big)+f\big(\dfrac{3+x}{1-x}\big)=x$ for all $x\in\mathbb{R}$, $x\neq -1$ and $x\neq 1$.

(Hint: Take first $y=\dfrac{x-3}{x+1}$ and then $y=\dfrac{3+x}{1-x}$. This will give two equations which are not difficult to solve.)

18. Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that $f\big(f(m)+f(n)\big)=m+n$ for all $m,n\in\mathbb{N}$.

(Hint: Show that $f(n$ is injective (one-to-one). Find $f(1)$.)

19. (Poland, 1992) Determine all functions $f:\mathbb{Q}^+\to\mathbb{Q}^+$ such that $f(x+1)=f(x)+1$ and $f(x^3)=\big(f(x)\big)^3$ for all $x\in\mathbb{Q}^+$.

(Hint: Consider the rational number $x=\dfrac{m}{n}+n^2$ for $m,n\in\mathbb{N}$.)

20. (Belarus, 1995) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f\big(f(x+y)\big)=f(x+y)+f(x)f(y)-xy$ for all $x,y\in\mathbb{R}$.

(Hint: This is a tricky one. One way of doing it is to try to get rid of the double f on the left hand side. You may first put $y=0$ and then replace x by $x+y$. Try the same trick with the new equation, but with $y=-1$ this time.)

21. (IMO, 1982) The function $f(n)$ is defined on the positive integers $\mathbb{N}$ and takes non-negative integer values. Moreover $f(2)=0$, $f(3)>0$, $f(9999)=3333$ and for all $m,n\in\mathbb{N}$: $f(m+n)-f(m)-f(n)=0$ or 1. Determine $f(1982)$.

(Hint: Since the condition $f(m+n)-f(m)-f(n)=0$ or 1 is not easy to handle we may try to replace it with a (weaker) condition $f(m+n)\geq f(m)+f(n)$. Find

$f(3)$ and then $f(3n)$.)

22. Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function satisfying $f(2) = 2$ and $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$ such that $(m, n) = 1$. (The notion (m, n) means the greatest common divisor of m and n. Thus, $(m, n) = 1$ means that m and n are coprime.)

Prove that $f(n) = n$ for all $n \in \mathbb{N}$.

(Hint: Show that if m is an odd integer and $f(m) = m$, then $f(2m) = 2m$. What is $f(3)$? Then, the indirect proof may be an effective method.)

23. (Chech Republic and Slovakia, 1993) Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(-1) = f(1)$ and $f(x) + f(y) = f(x + 2xy) + f(y - 2xy)$ for all $x, y \in \mathbb{Z}$.

(Hint: Find $f(3)$ and $f(5)$ in terms of $f(1)$. What pattern do you see? Show then that $f(n)$ is even. What can you find about the value $f(mn)$ for odd m?)

24. (IMO, 1977) The function $f(x)$ is defined on the set of positive integers and its values are positive integers. Given that $f(n+1) > f(f(n))$ for all n, prove that $f(n) = n$ for all $n \in \mathbb{N}$.

(Hint: Since we are given an inequality, it may turn out to be smart to stick to the inequalities and work on showing that $f(n) \geq n$ and $f(n) \leq n$. Consider as well proving the following statement: If $m \geq n$ then $f(m) \geq n$. This may be done by induction.)

25. Solve the same problem as in Example 19, but without the assumption that $f(x)$ must be continous.

(Hint: After beginning as in Example 19, show that $f(x + y) = f(x) + f(y)$ for non-constant solution $f(x)$. Prove then that $f(x) = x$ for all $x \in \mathbb{Q}^+$ and find then a way to extend the result to all $x \in \mathbb{R}^+$.)

26. (IMO, 2002) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$\big(f(x) + f(y)\big)\big((f(u) + f(v)\big) = f(xu - yv) + f(xv + yu)$ for all $x, y, u, v \in \mathbb{R}$.

(Hint: Find $f(x)$ for rational x and then try to extend the result to $x \in \mathbb{R}$. Since in the problem nothing is said about the continuity of $f(x)$, so you cannot use the standard argument. Instead you may find it useful to prove that $f(x)$ is even and monotone for $x \geq 0$.)

SOLUTIONS TO THE PROBLEMS OF THE COLLECTION

1. First we note that $x = 0$ must be excluded from the domain of $f(X)$. Substituting x by $-x$ yelds $-xf(-x) - 2xf(x) = -1$. Adding this equation twice to the original equation gives $-3xf(x) = -3$. Hence $f(x) = \frac{1}{x}$. It remains to verify that this function satisfy the given equation.

2. Replacing x by $\frac{1}{1-x}$ yelds the equqtion $f\Big(\frac{1}{1-x}\Big) + f\Big(\frac{x-1}{x}\Big) = \frac{1}{1-x}$. Replacing x again by $\frac{1}{1-x}$ gives $f\Big(\frac{x-1}{x}\Big) + f(x) = \frac{x-1}{x}$. Subtracting from this equation the previous one and adding the original equation yields $2f(x) = \frac{-x^3+x-1}{x(1-x)}$. Thus $f(x) = \frac{x^3-x+1}{2x(x-1)}$ and it is easy to verify that this function satisfy the original equation.

3. Let $y = \tan x$. Then $\sin 2x = \frac{2y}{y^2+1}$ and the equation can be written as $2f(y) + f(-y) = \frac{2y}{y^2+1}$.

Replacing now y with $-y$ gives a new equation $2f(-y) + f(y) = -\frac{2y}{y^2+1}$. If we now from this equation twice substract the first equation we get $-3f(y) = -\frac{6y}{y^2+1}$, i.e. $f(y) = \frac{2y}{y^2+1}$. Thus, $f(x) = \frac{2x}{x^2+1}$, and what remains is to check that this function satisfies the original equation.

4. Let $u = x + y$ and $v = x - y$. Then the equation can be written as $vf(u) - uf(v) = uv(u^2 - v^2)$. For $u \neq 0$ and $v \neq 0$ this can be written as $\frac{f(u)}{u} - u^2 = \frac{f(v)}{v} - v^2$.

Since this relation holds for arbitrary non-zero $u, v \in \mathbb{R}$ then $\frac{f(u)}{u} - u^2$ is constant. Thus, $\frac{f(u)}{u} - u^2 = c$, i.e. $f(x) = x^3 + cx$.

Observe that from the relation $vf(u) - uf(v) = uv(u^2 - v^2)$ follows (by taking $u = 0, v \neq 0$) that $f(0) = 0$. Since for each function $f(x) = x^3 + cx$ we have also

$f(0) = 0$ then, if the original equation has a solution, it must be $f(x) = x^3 + cx$ for any real constant c. Thus it only remains to check that this functions satisfy the given equation.

5. The equation can be written as $p(x+1) - (x+1)^2 = p(x) - x^2$, or, by letting $q(x) = p(x) - x^2$, as $q(x+1) = q(x)$. By induction one can show now that $q(x+n) = q(x)$ for all $n \in \mathbb{Z}$.

If we let $h(x) = q(x) - q(o)$, then it follows that $h(n) = 0$ for all $n \in \mathbb{Z}$. Since $h(x)$ is a polynomial then $h(x) \equiv 0$ and $q(x)$ is a constant polynomial. This implies that $p(x) = x^2 + c$, for any choice of a real constant c.

Substituting $p(x)$ in the original equation verifies that this family of polynomials satisfy the equation.

6. First one should find out that $p(0) = 0$ (by taking $x = 0$) and then that $p(k-1) = 0$ implies $p(k) = 0$ for $k = 1, 2, ..., 25$. Hence $p(x) = \prod_{k=0}^{25}(x-k) \cdot q(x)$, for some polynomial $q(x)$. Letting this expression for $p(x)$ into the original equaition yelds $x\prod_{k=0}^{25}(x-1-k) \cdot q(x-1) = (x-26)\prod_{k=0}^{25}(x-k) \cdot q(x)$.

It follows that for $x > 26$ we have $q(x-1) = q(x)$, and then, by the same argument as in Example 8, $q(x) = constant$. Finally $p(x) = c\prod_{k=0}^{25}(x-k)$, for any choice of the real constant c. It is easy to check that these polynomials satisfy the original equation.

7. In order to simplify the calculation let's introduce a new function $g(x) = f(x) - 1$. (This is a smart substitution allowing us to get rid of the constant -1 in the relation given in the problem). Replacing then $f(x)$ by $g(x) + 1$ in the relation gives ($\star$) $:\ g(xy) + g(x+y) = g(x)g(y) + g(x) + g(y)$ and $g(1) = 1$.

Inserting $y = 1$ in ($\star$) yelds $g(x) + g(x+1) = g(x)g(1) + g(x) + g(1)$, i.e. $g(x+1) = g(x) + 1$. Thus, for $x = 0$ we have $g(1) = g(0) + 1$, which means that $g(0) = 0$. Moreover, taking $x = -1$ we get $g(0) = g(-1+1) = g(-1) + 1$, i.e. $g(-1) = -1$.

By the induction we can now generalize the relation $g(x+1) = g(x) + 1$ to $g(x+n) = g(x) + n$ for all $n \in \mathbb{Z}$. Then it follows that $g(n) = g(0+n) = g(0) + n = n$ for all $n \in \mathbb{Z}$.

If we put $x = n$ and $y = \dfrac{1}{n}$ (for $0 \neq n \in \mathbb{Z}$) in the relation $(\star)$ then $g\big(\dfrac{n}{n}\big) + g\big(n+\dfrac{1}{n}\big) = g(n)g\big(\dfrac{1}{n}\big)+g(n)+g\big(\dfrac{1}{n}\big)$. Since $g(n) = n$ and $g\big(n+\dfrac{1}{n}\big) = n+g\big(\dfrac{1}{n}\big)$, then we have $1+n+g\big(\dfrac{1}{n}\big) = ng\big(\dfrac{1}{n}\big)+n+g\big(\dfrac{1}{n}\big)$, which implies that $g\big(\dfrac{1}{n}\big) = \dfrac{1}{n}$.

Let now $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Taking $x = m$ and $y = \dfrac{1}{n}$ in the relation $(\star)$ gives $g\big(\dfrac{m}{n}\big)+g\big(m+\dfrac{1}{n}\big) = g(m)g\big(\dfrac{1}{n}\big)+g(m)+g\big(\dfrac{1}{n}\big)$. This reduces to $g\big(\dfrac{m}{n}\big)+g(m)+g\big(\dfrac{1}{n}\big) = g(m)g\big(\dfrac{1}{n}\big) + g(m) + g\big(\dfrac{1}{n}\big)$, and further to $g\big(\dfrac{m}{n}\big) = \dfrac{m}{n}$.

Thus we have showed that $g(x) = x$ for all $x \in \mathbb{Q}$. Using the same continuity argument as in Example 9 we find that $g(x) = x$ for all $x \in \mathbb{R}$. Hence $f(x) = x+1$ and one should now verify that this function satisfy the relation in question.

8. Taking $y = 0$ yelds $xf(0) = xf(x^2)$ for all $x \in \mathbb{N}_0$, i.e. $f(x^2) = f(0)$. This may suggest that $f(x)$ is a constant function. Moreover, it is clear that all constant functions satisfy the given equation.

Suppose $a, b \in \mathbb{N}$ and $f(a) < f(b)$. Then $(a + b)f(a) = af(a) + bf(a) < af(b) + bf(a) < af(b) + bf(b) < (a + b)f(b)$. Since the middle term in the last expression equals $(a+b)f(a^2+b^2)$, then we have $(a+b)f(a) < (a+b)f(a^2+b^2) < (a + b)f(b)$, i.e. $f(a) < f(a^2 + b^2) < f(b)$ for all $a, b \in \mathbb{N}$.

We can then repeat the same argument with the same a and $b_1 = a^2 + b^2$ getting $f(a) < f(a^2 + b_1^2) < f(b_1) = f(a^2 + b^2) < f(b)$. Doing the same with $b_2 = a^2 + b_1^2$ we get an infinite number of different values $f(b_1), f(b_2), f(b_3), \ldots$, all of them between $f(a)$ and $f(b)$. Since this is imposible then, for all $a, b \in \mathbb{N}$, $f(a) = f(b)$. Especially $f(a) = f(1)$ for all $a \in \mathbb{N}$.

Since $f(1) = f(1^2) = f(0)$ then we finally have $f(x) = f(0)$ for all $x \in \mathbb{N}_0$.

9. Taking $x = 1$ and $y = 0$ yelds $f\big(f(0)\big) = 0$. Taking instead $x = 0$ yelds $f(y) = f\big(f(y)\big)$ for all $y \in \mathbb{R}$. Hence $f(0) = 0$.

By taking now $y = 0$ we get $f(x^4) = x^3 f(x)$. For $x \neq 0$ we have then $f(-x) = \dfrac{1}{(-x)^3}(-x)^3 f(-x) = -\dfrac{1}{x^3}f\big((-x)^4\big) = -\dfrac{1}{x^3}f(x^4) = -\dfrac{1}{x^3}x^3 f(x) = -f(x)$. Hence $f(x)$ is an odd function.

Suppose that $f(1) = 0$. Then we would have $f(2) = f(1 + 1) = f(1^4 + 1) = 1^3 f(1) + f\big(f(1)\big) = 0$, and, by induction, $f(n + 1) = f(1^4 + n) = 1^3 f(1) + f\big(f(n)\big) = 0$ for all $n \in \mathbb{N}$. This however cannot be the case since $f(x)$ only have a finite numbers of zeros. Hence $f(1) = c$ for some non-zero real constant c.

Puttning now $y = c$ in the expression $f(y) = f\big(f(y)\big)$ yelds $f(c) = f\big(f(c)\big) = f(1) = c$.

Suppose now that x_0 is a zero of $f(x)$, $x_0 \neq 0$. Then $x_0 \neq 1$ as well and, using the expression $f(x^4) = x^3 f(x)$, we get $f(x_0^4) = x_0^3 f(x_0) = 0$. That would give us an infinite number of zeros of $f(x)$, namely x_0^{4n} for all $n \in \mathbb{N}$. which is not possible. Thus $f(x_0) \neq 0$ for all $x_0 \neq 0$.

For any given $x \in \mathbb{R}$, let $z = f(x^4) - x^4$. Then we get $f(x^4) = f(x^4 + z) = x^3 f(x) + f\big(f(z)\big)$. At the same time $f(x^4) = x^3 f(x)$ according to one of the expressions above. Hence $f\big(f(z)\big) = 0$, which implies that $f(z) = 0$ and then $z = 0$.

From the last argument follows that $f(x) = x$ for all non-negative real numbers. But since $f(x)$ is an odd function then, for $x > 0$, we have $f(-x) = -f(x) = -x$. Thus $f(x) = x$ for all $x \in \mathbb{R}$. It is now easy to check that this function really satisfies the conditions of the problem.

10. By easy induction one can show that $f\Big(\prod_{i=1}^{k} a_i\Big) = \sum_{i=1}^{k} f(a_i)$. Now, we have $0 = f(10) = f(2 \cdot 5) = f(2) + f(5)$, and since $f(2), f(5) \geq 0$ then, $f(2) = f(5) = 0$.

In $n \in \mathbb{N}$ then we can factorize all 2's and 5's and write $n = 2^s \cdot 5^t \cdot m$, where the last (units) digit of m is 1, 3, 7 or 9. Hence, $f(n) = f(2^s \cdot 5^t \cdot m) = sf(2) + tf(5) + f(m) = f(m)$.

What remains is to find out what is $f(m)$ when the last digit of m is 1, 7 or 9. Suppose the last digit of m is 1, i.e. $m = 10k + 1$. Now we can use the second condition of the problem. We have $0 = f(3m) = f(3) + f(m) = f(m)$.

Similarily, if $m = 10k + 7$ then $0 = f(9m) = f(3) + f(3) + f(m) = f(m)$, and if $m = 10k + 9$ then $0 = f(3m) = f(3) + f(m) = f(m)$ (since the last digit of $3m$ is 7).

Thus $f(n) = 0$ for all $n \in \mathbb{N}$.

11. This is an easy problem and the solution may be like the following argument: $f(1, y) \cdot y \overset{(1)}{=} f(1, y) \cdot f(y, 1) \overset{(3)}{=} f(y, y) \overset{(2)}{=} 1$. Thus $f(1, y) = \dfrac{1}{y}$.

Now, $x \cdot \dfrac{1}{y} = x \cdot f(1, y) \overset{(1)}{=} f(x, 1) \cdot f(1, y) \overset{(3)}{=} f(x, y)$. Hence, $f(x, y) = \dfrac{x}{y}$.

It is obvious that this function satisfies given conditions.

12. Since $f(x) > 0$, it's safe to take logarithms on both sides. This yelds $2\ln\big(f(x)\big) = \ln\big(f(x+y)\big) + \ln\big(f(x-y)\big)$. The next step is obvious: introduce a new function: $g(x) = \ln\big(f(x)\big)$. The equation transforms to $2g(x) = g(x+y) + g(x-y)$.

In order to solve the new equation, take $y = x$. This gives $2g(x) = g(2x) + g(0)$, i.e. $g(2x) = 2g(x) - g(0)$. Now, since $2g(2x) = g(2x+x) + g(2x-x)$, then $g(3x) = 2g(2x) - g(x) = 2\big(2g(x) - g(0)\big) - g(x) = 3g(x) - 2g(0)$.

At this stage we may guess that $g(nx) = ng(x) - (n-1)g(0)$ for all $n \in \mathbb{N}$, and we may prove this by induction.

Take now a positive rational number $x = \dfrac{m}{n}$, with $n \in \mathbb{N}$. This means that $m = nx$ and thus, $g(m) = g(nx) = ng(x) - (n-1)g(0)$. On the other hand $g(m) = g(m \cdot 1) = mg(1) - (m-1)g(0)$.

From the last two equalities we deduce that $ng(x) - (n-1)g(0) = mg(1) - (m-1)g(0)$, i.e. $ng(x) = (n-m)g(0) + mg(1)$. Dividing both sides by n and keeping in mind that $x = \dfrac{m}{n}$, we get $g(x) = (1-x)g(0) + xg(1)$, which may be written as $g(x) = \big(g(1) - g(0)\big)x + g(0)$. Letting $g(1) - g(0) = a$ and $g(0) = b$ we get finally $g(x) = ax + b$.

The continuity of $g(x)$ (logarithm and $f(x)$ are continous) allow us to extend the result in the usual way to all $x \in \mathbb{R}^+$. Thus, $f(x) = e^{g(x)} = e^{ax+b}$ for any choice of real constants a and b.

What remains to do is to check that so obtained function satisfies the given equation.

13. It is obvious that the identity function $f(n) = n$ satisfies the given equation. We may suspect that there are no other functions than that.

First we observe that $f(n)$ is injective. For suppose $f(n) = f(m)$. Then obviously $f\big(f(n)\big) = f\big(f(m)\big)$ and consequently $f\Big(f\big(f(n)\big)\Big) = f\Big(f\big(f(m)\big)\Big)$. Thus, $f\Big(f\big(f(n)\big)\Big) + f\big(f(n)\big) + f(n) = f\Big(f\big(f(m)\big)\Big) + f\big(f(m)\big) + f(m)$, i.e. $3n = 3m$ and $n = m$.

For $n = 1$ we get $f\Big(f\big(f(1)\big)\Big) + f\big(f(1)\big) + f(1) = 3$, which can only mean that $f(1) = 1$. Hence $f(2) \geq 2$, $f(3) \geq 3$, and so on.

Suppose now that k is the least number such that $f(k) > k$. Then, since $f\big(f(k)\big) \geq f(k)$, we would have $f\big(f(k)\big) \geq f(k) > k$. Similarily we would have $f\Big(f\big(f(k)\big)\Big) \geq f\big(f(k)\big) > k$. This together would give us $f\Big(f\big(f(k)\big)\Big) + f\big(f(k)\big) + f(k) > 3k$ which contradicts the equation.

Hence $f(k) = k$ for all $k \in \mathbb{N}$

14. If the polynomial $p(x)$ is constant, $p(x) \equiv c$, then, inserting it into the equation gives $c = 0$ or $c = -1$. Both of this polynomials are apparently solutions to the equation. So let us now assume that $p(x)$ is not constant.

Suppose x_0 is a zero of $p(x)$. Putting x_0 into the equation yields $p(x_0^2) + p(x_0)p(x_0+1) = 0$, i.e. $p(x_0^2) = 0$. Thus x_0^2 is a zero of $p(x)$ as well. This argument can be repeated and, by induction, one shows that $x_0^{2^n}$ are zeros of $p(x)$ for all $n \in \mathbb{N}$. Since the polynomial $p(x)$ has only a finite number of zeros then x_0 can only equals 0, 1 or -1.

Letting now $x_0 - 1$ into the equation yields $p\big((x_0-1)^2\big) + p(x_0-1)p(x_0) = 0$, i.e. $p\big((x_0-1)^2\big) = 0$. This means that $(x_0-1)^2$ is again a zero of $p(x)$. In the view of the above discussion $(x_0-1)^2$ equals 0, 1 or -1. Hence, x_0 can only equals 0 or 1 and then $p(x) = cx^n(x-1)^m$ for some $c \in \mathbb{R}$ and $m, n \in \mathbb{N}$. If $c = 0$, we get the zero polynomial $p(x) \equiv 0$ already considered. Suppose then that $c \neq 0$

Inserting this expression into the equation gives $cx^{2n}(x^2-1)^m + cx^n(x-1)^m \cdot c(x+1)^n x^m = 0$, which reduces to $x^{n-m}(x+1)^{m-n} + c = 0$ for all x. Then apparently $m = n$ and $c = -1$. Hence $p(x) = -x^n(x-1)^n$ for all $n \in \mathbb{N}$.

One must now only check that these functions really satisfy the given equation. Thus the answer is $p(x) \equiv 0$ or $p(x) \equiv -1$ or $p(x) = -x^n(x-1)^n$ for all $n \in \mathbb{N}$.

15. Suppose $r, s \in \mathbb{Q}$ such that $r < s$, and let n be a positive integer. Let divide the segment $[r, s]$ in n equal parts by $r_i = r + \dfrac{s-r}{n} \cdot i$, for $i = 0, 1, 2, ..., n$. Each part has the length $|r_i - r_{i+1}| = \dfrac{s-r}{n}$.

Then $|f(r) - f(s)| = \left|\sum_{i=0}^{n-1}\Big(f(r_i) - f(r_{i+1})\Big)\right|$. Using now the triangle inequality we get $|f(r) - f(s)| \leq \sum_{i=0}^{n-1}\Big|f(r_i) - f(r_{i+1})\Big| \leq 7\sum_{i=0}^{n-1}\big(r_i - r_{i+1}\big)^2 = 7\sum_{i=0}^{n-1}\Big(\dfrac{s-r}{n}\Big)^2 = \dfrac{7(r-s)^2}{n}$.

Letting now $n \to \infty$ we find out that the right hand side goes to 0 and so $f(r) - f(s) = 0$, i.e. $f(r) = f(s)$ for all $r, s \in \mathbb{Q}$. Hence $f(x)$ is a constant function on $\mathbb{Q}$, $f(x) = c$ for some real constant c and all $x \in \mathbb{Q}$.

Now we can turn to the real numbers x. If then $x \in \mathbb{R}$ wa may consider a sequence of rational numbers $\{r_n\} \subset \mathbb{Q}$, such that $\lim_{n\to\infty} r_n = x$. We may in fact choose $\{r_n\}$ so that $|x - r_n| < 10^{-n}$ for all $n \in \mathbb{N}$. Thus

$$|f(x) - c| = |f(x) - f(r_n)| \leq 7(x - r_n)^2 < \frac{7}{10^{2n}}.$$

Since the right-hand side can be made as small as needed, we conclude that $f(x) - c = 0$. Hence $f(x) = c$ for all $x \in \mathbb{R}$. It is also easy to see that the constant function really satisfy the given equation.

16. Taking $\frac{1}{x}$ instead of x yelds $a\frac{1}{x^2}f(x) + f\left(\frac{1}{x}\right) = \frac{1}{x+1}$. Eliminating now $f\left(\frac{1}{x}\right)$ from this equation and the original one reduces to $(1 - a^2)f(x) = \frac{x(1 - ax)}{x+1}$.

Consider now several cases depending on the constant a:

(1) If $a = -1$ or $a = 1$, then we have $0 = \frac{x(1 - ax)}{x+1}$ for all $x > 0$. This is clearly impossible, hence there is no solution in this case.

(2) If $a > 1$, then $f(x) = \frac{1}{1 - a^2} \cdot \frac{x(1 - ax)}{x+1}$. Taking then $x = \frac{1}{2a}$ we will get $f\left(\frac{1}{2a}\right) < 0$, which contradicts the condition on $f(x)$ ($f(x)$ is positive valued).

(3) If $a < -1$, then again $f(x) = \frac{1}{1 - a^2} \cdot \frac{x(1 - ax)}{x+1}$ and $f(x) < 0$ for all $x > 0$. Hence a contradiction.

(4) If $0 < a < 1$, then, since $f(x) = \frac{1}{1 - a^2} \cdot \frac{x(1 - ax)}{x+1}$, we have $f(x) < 0$ for all $x > \frac{1}{a}$. Again a contradiction.

(5) If $-1 < a < 0$, then $f(x) = \frac{1}{1 - a^2} \cdot \frac{x(1 - ax)}{x+1}$ and this is > 0 for all positive real numbers x.

It remains to verify that this function (only for $-1 < a < 0$) satisfy the given equation.

17. Taking $y = \frac{x - 3}{x + 1}$ yelds $x = \frac{3 + y}{1 - y}$ and the equation $f(y) + f\left(\frac{y - 3}{y + 1}\right) = \frac{3 + y}{1 - y}$. If we instead take $y = \frac{3 + x}{1 - x}$ then $x = \frac{y - 3}{y + 1}$ and we get another equation: $f\left(\frac{3 + y}{1 - y}\right) + f(y) = \frac{y - 3}{y + 1}$.

Adding now both equations together we get $2f(y) + f\left(\frac{y - 3}{y + 1}\right) + f\left(\frac{3 + y}{1 - y}\right) =$

$\dfrac{3+y}{1-y}+\dfrac{y-3}{y+1}=\dfrac{8y}{1-y^2}$. On the other hand we know from the original functional equation that $f\Big(\dfrac{y-3}{y+1}\Big)+f\Big(\dfrac{3+y}{1-y}\Big)=y$. Hence $2f(y)+y=\dfrac{8y}{1-y^2}$. Finally $f(y)=\dfrac{4y}{1-y^2}-\dfrac{y}{2}=\dfrac{y^3+7y}{2(1-y^2)}$.

It remains to verify that this function really satisfy the given equation.

18. Suppose $f(m)=f(n)$. Then we have $f(m)+f(n)=f(n)+f(n)$. Taking f on both sides of last equality gives, according to the relation the function f satisfies, $m+n=f\big(f(m)+f(n)\big)=f\big(f(n)+f(n)\big)=n+n$. Hence $m=n$ and we can conclude that $f(n)$ is injective.

In order to find the value $f(1)$ suppose $f(1)=c>1$. Then $2=f\big(f(1)+f(1)\big)=f(2c)$. Thus $f(2+c)=f\big(f(2c)+f(1)\big)=2c+1$. It is then obvious that c cannot equals 2: putting $c=2$ into the last two equalities would give $f(4)=2$ and $f(4)=5$. Hence $c>2$.

Consider now the numbers $f(2c)+f(1)$ and $f(c+2)+f(c-1)$. Applying f to those two numbers yields $f\big(f(2c)+f(1)\big)=2c+1$ and $f\big(f(c+2)+f(c-1)\big)=c+2+c-1=2c+1$. Since $f(n)$ is injective then $f(2c)+f(1)=f(c+2)+f(c-1)$, which means that $2+c=1+2c+f(c-1)$, i.e. $f(c-1)=1-c<0$. Since this is impossible then the laternative $c>2$ must be rejected and we have $c=1$.

Now we claim that $f(n)=n$ for all $n\in\mathbb{N}$. We know it is true for $n=1$. So suppose it is true for some $n_0\in\mathbb{N}$. Then $n_0+1=f\big(f(n_0)+f(1)\big)=f(n_0+1)$. Hence, by the induction, $f(n)=n$ for all $n\in\mathbb{N}$.

19. By easy induction one may extend the condition $f(x+1)=f(x)+1$ to $f(x+n)=f(x)+n$ for all $n\in\mathbb{N}$.

Consider now the positive rational number $x=\dfrac{m}{n}+n^2$, for any $m,n\in\mathbb{N}$. From the second condition of the problem and using the new condition above we get $f\Big((\dfrac{m}{n}+n^2)^3\Big)=\Big(f(\dfrac{m}{n}+n^2)\Big)^3=\Big(f(\dfrac{m}{n})+n^2\Big)^3=\Big(f(\dfrac{m}{n})\Big)^3+3\Big(f(\dfrac{m}{n})\Big)^2n^2+3f(\dfrac{m}{n})n^4+n^6$.

On the other hand $f\Big((\dfrac{m}{n}+n^2)^3\Big)=f\Big((\dfrac{m}{n})^3+3(\dfrac{m}{n})^2n^2+3(\dfrac{m}{n})n^4+n^6\Big)=f\Big((\dfrac{m}{n})^3+3m^2+3mn^3+n^6\Big)=f\Big((\dfrac{m}{n})^3\Big)+3m^2+3mn^3+n^6=\Big(f(\dfrac{m}{n})\Big)^3+$

$3m^2+3mn^3+n^6$.

Equating both right-hand sides gives $\Big(f(\frac{m}{n})\Big)^3+3\Big(f(\frac{m}{n})\Big)^2 n^2+3f(\frac{m}{n})n^4+n^6=\Big(f(\frac{m}{n})\Big)^3+3m^2+3mn^3+n^6$, i.e. $\Big(f(\frac{m}{n})\Big)^2 n^2+f(\frac{m}{n})n^4=m^2+mn^3$.

Now, it is only to discover thet the last expression can be factorised as $0=\Big(f(\frac{m}{n})\Big)^2 n^2+f(\frac{m}{n})n^4-m^2-mn^3=\Big(f(\frac{m}{n})n-m\Big)\Big(f(\frac{m}{n})n+m+n^3\Big)$.

Since the last parenthesis is never 0, then $f(\frac{m}{n})n-m=0$, which means that $f(\frac{m}{n})=\frac{m}{n}$.

It is not difficult that the function $f(x)=x$ satisfies the original equation.

20. It is obvious thet the function $f(x)$ is not constant. The double f of the left hand side complicates the problem considerably. To get rid of that we may first take $y=0$ (getting $f\big(f(x)\big)=f(x)+f(x)f(0)$) and then replace x by $x+y$, which results in a new equation $f\big(f(x+y)\big)=f(x+y)+f(x+y)f(0)$. Equating the right-hand sides of this equation and the original one we get $f(x+y)+f(x)f(y)-xy=f(x+y)+f(x+y)f(0)$, or $f(0)f(x+y)=f(x)f(y)-xy$ ($\star$).

Let's now try to put $y=1$ into ($\star$). This will result in $f(0)f(x+1)=f(x)f(1)-x$ ($\star\star$). From the last expression we would like to eliminate the $f(x+1)$ term. In order to do that put $y=-1$ in ($\star$) (getting $f(0)f(x-1)=f(x)f(-1)+x$) and replace then x by $x+1$. This gives $f(0)f(x)=f(x+1)f(-1)+x+1$, which multiplied by $f(0)$ is $f^2(0)f(x)=f(0)f(x+1)f(-1)+f(0)(x+1)$. Now we can substitute here $f(0)f(x+1)$ by the expression in ($\star\star$): $f^2(0)f(x)=\big(f(x)f(1)-x\big)f(-1)+f(0)(x+1)$. Hence $\Big(f^2(0)-f(1)f(-1)\Big)f(x)=\Big(f(0)-f(-1)\Big)x+f(0)$ ($\star\star\star$).

There are now two cases to consider: the coefficient on the left hand side equals 0 or not.

Suppose $f^2(0)-f(1)f(-1)=0$. Then putting $x=0$ in ($\star\star\star$) results in $f(0)=0$. Hence, $f(1)f(-1)=0$. At the same time the equality $f(0)=0$ turn ($\star$) into $f(x)f(y)=xy$. Taking $x=1$ and $y=-1$ we get $f(1)f(-1)=-1$, which contradicts the previous result.

Suppose finally that $f^2(0)-f(1)f(-1)\neq 0$. Then the expression ($\star\star\star$) implies that $f(x)$ is a polynomial of degree one, $f(x)=ax+b$. Substituting this polynomial into the original equation we get $a\big(a(x+y)+b\big)+b=a(x+y)+$

$b + (ax + b)(ay + b) - xy$. Since this is valid for all $x, y \in \mathbb{R}$ the by taking some values for x and y it is easy now to show that $a = 1$ and $b = 0$. Thus, the only solution to the equation is $f(x) = x$.

21. Since the condition $f(m + n) - f(m) - f(n) = 0$ or 1 is not easy to handle we may try to replace it with a (weaker) condition $f(m + n) \geq f(m) + f(n)$. So let's see how far do we get.

We begin with finding $f(1)$: $0 = f(2) = f(1 + 1) \geq f(1) + f(1) = 2f(1)$. Since $f(1) \geq 0$ the we have $f(1) = 0$.

Now, $f(3) = f(2 + 1) = f(2) + f(1) + a = a$, where a equals 0 or 1. Since we know that $f(3) > 0$ then, of course, $f(3) = 1$

Next we may note that $f(2 \cdot 3) = f(3 + 3) \geq f(3) + f(3) = 2$, $f(3 \cdot 3) = f(2 \cdot 3 + 3) \geq f(2 \cdot 3) + f(3) \geq 3$, and generally, by induction, that $f(3 \cdot n) \geq n$.

Moreover, if we for some k get $f(3k) > k$, then the same argument shows that $f(3m) > m$ for all $m > k$. But we know that $f(9999) = f(3 \cdot 3333) = 3333$, hence $f(3n) = n$ for all n upp to at least 3333.

Now, $1982 = f(3 \cdot 1982) = f(2 \cdot 1982 + 1982) \geq f(2 \cdot 1982) + f(1982) = f(1982 + 1982) + f(1982) \geq 3f(1982)$, implying that $f(1982) \leq 660$. On the other hand, $f(1982) = f(1980 + 2) \geq f(1980) + f(2) = f(3 \cdot 660) = 660$. Thus, $f(1982) = 660$.

22. It is obvious that $f(1) = 1$. If m is an odd integer then $(m, 2) = 1$ and we have $f(2m) = f(2)f(m) = 2f(m)$. Hence, if m is an odd integer and $f(m) = m$, then $f(2m) = 2m$.

Let's try to find the value of $f(3)$. There are many ways of doing this, for example through the following, rather artificial, reasoning (remember that $f(x)$ is strictly increasing):

$2f(7) = f(2)f(7) < f(3)f(7) = f(21) < f(22) = f(2)f(11) = 2f(12) < 2f(14) = 2f(2)f(7) = 4f(7)$. Thus, $2f(7) < f(3)f(7) < 4f(7)$, giving $2 < f(3) < 4$, i.e. $f(3) = 3$.

Suppose now there are some positive integers n for which $f(n) \neq n$. Let then n_0 be the smallest among them. We have, of course, $n_0 > 3$ and for all n such that $1 \leq n < n_0$, $f(n) = n$.

From this it follows that $f(n_0) > n_0$ and morover, since $f(x)$ is strictly increasing, $f(n) > n$ for all $n \geq n_0$ $(\star)$.

Let now then consider two cases:

(1) If n_0 is odd then $(2, n_0 - 2) = 1$ and so $2(n_0 - 2) = f(2)f(n_0 - 2) =$

$f\big(2(n_0-2)\big)$. However, for $n_0 > 3$, $2(n_0-2) \geq n_0$ and so, according to ($\star$), $f\big(2(n_0-2)\big) > 2(n_0-2)$, giving $2(n_0-2) > 2(n_0-2)$, thus a contradiction.

(2) If n_0 is even then $(2, n_0-1) = 1$ and so $2(n_0-1) = f(2)f(n_0-1) = f\big(2(n_0-1)\big)$. Again, for $n_0 > 3$, $2(n_0-1) \geq n_0$ and so $f\big(2(n_0-1)\big) > 2(n_0-1)$, giving $2(n_0-1) > 2(n_0-1)$, a contradiction.

In conclusion, such n_0 doesn't exist and hence $f(n) = n$ for all $n \in \mathbb{N}$.

23. Let's try some values of x and y. Taking $x = y = 1$ yields $f(1) + f(1) = f(3) + f(-1)$, but since $f(-1) = f(1)$ then $f(3) = f(1)$.

Taking now $x = 1$ and $y = 2$ gives $f(1) + f(2) = f(5) + f(-2)$, but if we take $x = 2$ and $y = -1$ then $f(2) + f(-1) = f(-2) + f(3)$. From the two last expression we find that $f(5) = f(3)$.

Since then $f(1) = f(3) = f(5)$ we may suspect that $f(n)$ has the same value for all odd integers.

This is in fact correct and in order to prove it just take first $x = 1$ and $y = m$ (giving $f(1) + f(m) = f(1+2m) + f(-m)$) and then $x = m$ and $y = -1$ (so we get $f(m) + f(-1) = f(-m) + f(-1+2m)$. From the two last expression we find that $f(2m-1) = f(2m+1)$ for all $m \in \mathbb{Z}$.

Another consequence of the equation $f(1) + f(m) = f(1+2m) + f(-m)$) is now that $f(m) = f(-m)$. Since then $f(x)$ ie an even function, it is suficcient to find the expression for $f(x)$ for non-negative integers x.

Let now $x = n$ and $y = -(2k+1)$. Then our equation implies $f(n) + f\big(-(2k+1)\big) = f\big(-n(1+4k)\big) + f\big(-(2k+1)(1-2n)\big)$, which, after cancelling $f(m)$ for odd m, means that $f(n) = f\big(-n(1+4k)\big) = f\big((1+4k)n\big)$. If we instead take $x = -(2k+1)$ and $y = n$ then $f\big(-(2k+1)\big) + f(n) = f\big(-(2k+1)(1+2n)\big) + f\big(n(4k+3)\big)$, i.e. $f(n) = f\big((4k+3)n\big)$. Thus $f(n) = f(mn)$ for any odd ineger m.

Every positive integer n can be written in form $n = 2^a m$ for some non-negative integer a and an odd integer m. Hence $f(n) = f\big(2^a m\big) = f\big(2^a\big)$. Thus any function with of the kind we are looking for is determined by the values $f(0)$, $f(1)$, $f(2)$, $f\big(2^2\big)$, $f\big(2^3\big)$, $f\big(2^4\big)$ and so on, which may all be chosen arbitrary. All other values are given by $f(n) = f\big(2^a\big)$ as above. For negative integers k we have that $f(k) = f(-k)$.

Finally let's check that such functions satisfy the equations. Clearly $f(-1) = f(1)$. If $x = 0$ or $y = 0$ then the equation becomes an identity. So suppose that $x = 2^a m$ and $y = 2^b n$ for some non-negative integers a and b and odd m, n. Then the left-hand side of the equation becomes $f(2^a m) + f(2^b n) = f(2^a) + f(2^b)$, while the right-hand side becomes $f\big(2^a m(1+2y)\big) + f\big(2^b n(1-2x)\big) = f(2^a) + f(2^b)$, since both $m(1+2y)$ and $n(1-2x)$ are odd.

Thus the solution is complete.

24. Since $f : \mathbb{N} \to \mathbb{N}$, it ishould be clear that $f(1) > 1$. Moreover, $f(2) > f(f(1)) \geq 1$, which implies that $f(2) \geq 2$. The same argument cannot however be extended for showing that $f(3) \geq 3$.

Nevertheless it is possible to prove slightly stronger statement, from which the inequality $f(n) \geq n$ follows immediately.

Statement: If $m \geq n$ then $f(m) \geq n$.

This statement is obviously true for $n = 1$ since $f(m) \geq 1$ by the definition of $f(m)$. So let us assume that the statement is true for some $n_0 \geq 1$, i.e. assume that If $m \geq n_0$ then $f(m) \geq n_0$.

Now, suppose that $m \geq n_0 + 1$. hence, $m - 1 \geq n_0$ and then, by the assumption, $f(m-1) \geq n_0$. By the assumption again, $f\big(f(m-1)\big) \geq n_0$. Using the property of $f(m)$ in the statment of the problem, we know that $f(m) > f\big(f(m-1)\big)$. Hence $f(m) > f\big(f(m-1)\big) \geq n_0$. This means that $f(m) \geq n_0 + 1$ and, by induction, the statment is true for all $n \in \mathbb{N}$.

As a special case, we have $f(n) \geq n$ for all $n \in \mathbb{N}$. From this it follows that $f(n+1) > f(f(n)) \geq f(n)$, proving that the function $f(n)$ is strictly increasing.

Finally, suppose that $f(n) \neq n$ for some $n \in \mathbb{N}$. Then, of course, $f(n) > n$ and we get $f(n) \geq n + 1$. This implies that $f(n+1) > f(f(n)) \geq f(n+1)$, which is impossible.

Therefore, $f(n) = n$ for all $n \in \mathbb{N}$.

25. As in the solution of Example 19, we discover that the constant function $f_0(x) \equiv 1$ is a solution to the equation, and then that for non-constant solution $f(x)$ we have $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}^+$.

Now we find that $f(a)^{f(x+y)} = f\big(a^{x+y}\big) = f\big(a^x a^y\big) =$ (by the previous equality) $= f\big(a^x\big)f\big(a^y\big) = f(a)^{f(x)} f(a)^{f(y)} = f(a)^{f(x)+f(y)}$, from which follows that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^+$.

By the same methods as in Example 9 one can show now that $f(x) = xf(1)$ for all $x \in \mathbb{Q}^+$. At the same time, setting $x = y = 1$ into the equation $f(xy) = f(x)f(y)$, we get $f(1) = 1$, and thus $f(x) = x$ for all $x \in \mathbb{Q}^+$. What then remains is to extend this formula to all $x \in \mathbb{R}^+$.

Suppose that for some $x > 0$ we have $f(x) < x$ (in the case $f(x) > x$ the argument is similar). Let's pick up a rational number a such that $f(x) < a < x$. Then we will have $f(x) = f\bigl(a + (x - a)\bigr) = f(a) + f(x - a) > f(a) = a$, which contradicts the choice of a. Hence $f(x) \equiv 1$ and $f(x) = x$ are the only solutions to the problem.

26. By letting $x = y = 0$ and $u = v$, we get $4f(0)f(u) = 2f(0)$. So either $f(u) = 1/2$ for all $u \in \mathbb{R}$, or $f(0) = 0$. The constant function $f(u) = 1/2$ is certainly a solution. Hence assume that $f(0) = 0$.

Putting $y = v = 0$ we get $f(x)f(u) = f(xu)$ ($\star$). In particular, taking $x = u = 1$, we have $f(1)^2 = f(1)$. Hence $f(1) = 0$ or $f(1) = 1$. Suppose that $f(1) = 0$. By taking $x = y = 1$ and $v = 0$, we get $0 = 2f(u)$. Thus $f(x) = 0$ for all $u \in \mathbb{R}$. That is certainly a solution as well. We can thus assume that $f(1) = 1$.

Setting $x = 0$ and $u = v = 1$, we get $2f(y) = f(y) + f(-y)$, which reduces to $f(-y) = f(y)$. This means that $f(x)$ is an even function and so we need only consider $f(x)$ for positive x.

Next we show that $f(r) = r^2$ for all $r \in \mathbb{Q}$. The first step is to show that $f(n) = n^2$ for all $n \in \mathbb{N}$. This is done by the induction on n. It is obviously true for $n = 0$ and 1. Suppose it is true for $n - 1$ and n. Then letting $x = n$ and $y = u = v = 1$ into the equation, we get $2f(n) + 2 = f(n - 1) + f(n + 1)$. Hence $f(n + 1) = 2n^2 + 2 - (n - 1^)2 = (n + 1)^2$. Hence the statment is true for $n + 1$.

Now the relation ($\star$) implies that $f(n)f\bigl(\frac{m}{n}\bigr) = f(m)$, so $f\bigl(\frac{m}{n}\bigr) = \frac{m^2}{n^2}$ for all $m, n \in \mathbb{N}$. Hence we have established that $f(r) = r^2$ for all $r \in \mathbb{Q}^+$. By the fact that $f(x)$ is even, $f(r) = r^2$ for all $r \in \mathbb{Q}$.

Now it is natural to suspect that $f(x) = x^2$ for all $x \in \mathbb{R}$, so this is what we should try to prove in the final step. Since we don't have the condition that $f(x)$ is continous, we cannot make use of the standard procedure for those cases.

From the relation ($\star$) above, we have $f(x^2) = f(x)^2 \geq 0$, so $f(x^2)$ is always non-negative. Hence $f(x) \geq 0$ for positive x and, again by the fact that $f(x)$ is even, $f(x) \geq 0$ for all $x \in \mathbb{R}$.

Putting now $u = y$ and $v = x$, we get $\bigl(f(x) + f(y)\bigr)^2 = f(x^2 + y^2)$, so $f\bigl(x^2 + y^2\bigr) = f(x)^2 + 2f(x)f(y) + f(y)^2 \geq f(x)^2 = f(x^2)$. For any u and v such that $u > v > 0$, we may put $v = x^2$ and $u = x^2 + y^2$, and hence $f(u) \geq f(v)$. In other words, $f(x)$ is an increasing function.

Thus for any real x we may take a sequence of rationals r_n, all less than x, that converge to x and another sequence of rationals s_n, all greater than x, which also converge to x. Then we get $r_n^2 = f(r_n) \leq f(x) \leq f(s_n) = s_n^2$ for all $x \in \mathbb{R}$ and hence $f(x) = x^2$.

The final answer is then: there are three possible functions solutions, namely $f(x) = 0$ for all $x \in \mathbb{R}$, $f(x) = \dfrac{1}{2}$ for all $x \in \mathbb{R}$ or $f(x) = x^2$.

ADDITIONAL PROBLEMS

Here follows some more problems, this time without solutions offered. Instead, after the problems there are some hints and answers.

Problems.

27. (Poland, 1992) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) - f(x-y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

28. Find all functions that satisfy the equation $f(1-x) + xf(x-1) = \dfrac{1}{x}$ for all real $x \neq 0$, $x \neq 1$ and $x \neq -1$.

29. Find all continous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation $f(x+y) = f(x) + f(y) + xy$ for all $x, y \in \mathbb{R}$.

30. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satsifying $xf(y) + yf(x) = (x+y)f(x)f(y)$ for all $x, y \in \mathbb{R}$.

31. Find all injective (one-to-one) functions $f : \mathbb{R} \to \mathbb{R}$ such that $f\big(f(x) + y\big) = f(x+y) + 1$ for all $x, y \in \mathbb{R}$.

32. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying following conditions $f(1) = 1$ and $f(x+y)\big(f(x) - f(y)\big) = f(x-y)\big(f(x) + f(y)\big)$ for all $x, y \in \mathbb{Z}$.

33. Find all polynomials $p(x)$ satisfying the equation $p(x^2 - 2x) = \big(p(x-2)\big)^2$ for all $x \in \mathbb{R}$.

34. Find all continous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 1$ and $f\sqrt{x^2 + y^2} = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

35. Find all functions defined for $x > 0$, such that $xf(y) + yf(x) = f(xy)$ for all $x, y \in \mathbb{R}^+$.

36. Find all continous solutions $f : \mathbb{R} \to \mathbb{R}$ to the equation $f(x+y) - f(x-y) = f(x)$ for all $x, y \in \mathbb{R}$.

37. Find all functions $f : \mathbb{R} \to \mathbb{R}$, continous in $x = 0$ which satisfy the equation $f(x + y) = f(x) + f(y) + xy(x + y)$ for all $x, y \in \mathbb{R}$. (Compare with problem 25.)

38. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which solve the equation $f(x+y)+f(x-y) = 2f(x)\cos y$ for all $x, y \in \mathbb{R}$.

39. Suppose $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasin function such that $f(f(n)) = 3n$ for all $n \in \mathbb{N}$. Find all possible values of $f(1977)$.

40. (AMM, Problem E2176) Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(2) = 2$ and $f\Big(\dfrac{x + y}{x - y}\Big) = \dfrac{f(x) + f(y)}{f(x) - f(y)}$ for all rational $x \neq y$.

41. (Austria-Poland, 1997) Show that there is no function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f\big(x + f(y)\big) = f(x) - y$ for all $x, y \in \mathbb{Z}$.

42. (Ukraine, 1997) Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $f(x + 1) = f(x) + 1$ and $f(x^2) = \big(f(x)\big)^2$ for all $x \in \mathbb{Q}^+$.

43. (IMO short-list, 1999) Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies two conditions: $|f(x)| \leq 1$ for all $x \in \mathbb{R}$ and $f\big(x+\frac{13}{42}\big)+f(x) = f\big(x+\frac{1}{6}\big)+f\big(x+\frac{1}{7}\big)$ for all $x \in \mathbb{R}$. Prove that $f(x)$ is periodic.

44. (IMO, 1981) The function $f(x, y)$ satisfies:
$f(0, y) = y + 1, f(x + 1, 0) = f(x, 1)$ and $f(x + 1, y + 1) = f(x, f(x + 1, y))$ for all non-negative integers x, y. Find $f(4, 1981)$.

45. (IMO, 2004) Find all polynomials $P(x)$ with real coefficients which satisfy the equality

$$P(a-b)+P(b-c)+P(c-a)=2P(a+b+c)$$

for all real numbers a, b, c such that $ab+bc+ca=0$.

46. (IMO, 1994) Let S be the set of all real numbers greater than -1. Find all functions $f\colon S \to S$ such that $f\big(x+f(y)+xf(y)\big)=y+f(x)+yf(x)$ for all $x, y \in S$, and $\dfrac{f(x)}{x}$ is strictly increasing on each of the intervals $-1<x<0$ and $0<x$.

47. (IMO, 1992) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f\big(x^2+f(y)\big)=y+\big(f(x)\big)^2$ for all $x, y \in \mathbb{R}$.

48. (Iran, 1999) Suppose $f:\mathbb{R}^+\to\mathbb{R}^+$ is a strictly decreasing function which satisfy the equation

$$f(x+y)+f\big(f(x)+f(y)\big)=f\Big(f\big(f(x)+y\big)+f\big(x+f(y)\big)\Big) \text{ for all } x, y \in \mathbb{R}^+.$$

Prove that $f\big(f(x)\big)=x$ for all $x\in\mathbb{R}^+$.

49. (IMO, 1988) A function f is defined on the positive integers $\mathbb{N}$ by:

$f(1)=1, f(3)=3, f(2n)=f(n), f(4n+1)=2f(2n+1)-f(n)$, and
$f(4n+3)=3f(2n+1)-2f(n)$ for all $n\in\mathbb{N}$.

Determine the number of positive integers $n\le 1988$ for which $f(n)=n$.

50. (IMO, 1986) Find all functions $f(x)$ defined on the non-negative real numbers and taking non-negative real values such that: $f(2)=0$, $f(x)\ne 0$ for $0\le x<2$, and $f\big(xf(y)\big)f(y)=f(x+y)$ for all non-negative real x, y.

51. (IMO, 1998) Consider all functions $f:\mathbb{N}\to\mathbb{N}$ satisfying $f\big(t^2f(s)\big)=s\big(f(t)\big)^2$ for all $s, t\in\mathbb{N}$. Determine the least possible value of $f(1998)$.

52. (IMO, 1990) Construct a function $f:\mathbb{Q}^+\to\mathbb{Q}^+$ such that $f\big(xf(y)\big)=\dfrac{f(x)}{y}$ for all $x, y\in\mathbb{Q}^+$.

Hints and answers.

27. Hint: Try some values.

Answer: $f(x) \equiv 0$.

28. Hint: Create a new equation.

Answer: $f(x) = \dfrac{3-x}{x(1-x^2)}$.

29. Hint: One way of solving is the standard procedure, finding $f(n)$ for integers n, extend it to $\mathbb{Q}$ and then to $\mathbb{R}$.

Another way is to guess that $f(x) = \dfrac{1}{2}x^2$ is one solution. Are there more?

Answer: $f(x) = \frac{1}{2}x^2 + cx$ for any real constant c.

30. Hint: Try some values.

Answer: $f(x) \equiv 0$ or $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ c & \text{if } x = 0 \end{cases}$ for any real constant c.

31. Hint: Let x and y change places.

Answer: $f(x) = x + 1$.

32. Hint: Show that $f(n)$ is odd. Take then $x = 2$ and $y = 1$ and consider some cases.

Answer: $f(n) = \begin{cases} 0 & \text{if } n = 2k \\ 1 & \text{if } n = 4k+1 \\ -1 & \text{if } n = 4k+3 \end{cases}$, or $f(n) = \begin{cases} 0 & \text{if } n = 3k \\ 1 & \text{if } n = 3k+1 \\ -1 & \text{if } n = 3k+2 \end{cases}$,

for $k \in \mathbb{Z}$, or $f(n) = n$ for all $n \in \mathbb{Z}$.

33. Hint: Solve first the functional equation $q(x^2) = \big(q(x)\big)^2$, where $q(x)$ is a polynomial.

Answer: $p(x) \equiv 0$ or $p(x) = (x+1)^n$ for each $n \in \mathbb{N}$.

34. Hint: Find the expression for $f(x)$ first for x in $\mathbb{N}$, then in $\mathbb{Q}$.

Answer: $f(x) = x^2$.

35. Hint: Transform to a Cauchy-type equation.

Answer: $f(x) = cx \ln x$ for any real constant c.

36. Hint: Standard procedure.

Answer: $f(x) = cx$ for any real constant c.

37. Hint: One way of solving is the standard procedure, finding $f(n)$ for integers n, extend it to $\mathbb{Q}$ and then to $\mathbb{R}$. But first you will have to show that the continuty in $x = 0$ will imply that $f(x)$ is continous for all $x \in \mathbb{R}$.

Another way is to discover that $3xy(x+y)$ is a part of the expression for $(x+y)^3$ and thus guess that $f(x) = \dfrac{1}{3}x^3$ is one solution. Are there more?

Answer: $f(x) = \frac{1}{3}x^3 + cx$ for any real constant c.

38. Hint: Start with some values for x and y.

Answer: $f(x) = a\cos x + b\sin x$ for any choice of $a, b \in \mathbb{R}$.

39. Hint: Prove that $f(3k) = 3f(k)$. Show then that for $3^m \leq n < 2 \cdot 3^m$ the function is $f(n) = n + 3^m$, while for $2 \cdot 3^m \leq n < 3^{m+1}$ one must have $f(n) = 3n - 3^{m+1}$.

Answer: $f(1997) = 3804$.

40. Hint: Find $f(0)$ and $f(1)$. Show then that $f\big(\dfrac{m}{n}\big) = \dfrac{f(m)}{f(n)}$ for all $m, n \in \mathbb{Z}$, $n \neq 0$.

Answer: $f(x) = x$.

41. Hint: Show that $g(x) = f\big(f(x)\big)$ is injective (in fact linear) and then show that $f(x)$ satisfy the first of Cauchy's equations. Find then the contradiction to the existens of the solution.

42. Hint: Count $f\Big(\big(\dfrac{m}{n} + n\big)^2\Big)$ in two different ways.

Answer: $f(x) = x$.

43. Hint: Show that the function $g(x) = f(x + \dfrac{1}{6}) - f(x)$ is periodic and then that the function $h(x) = f(x+1) - f(x)$ is periodic.

Answer: The sortest period for $f(x)$ is 1.

44. Hint: Calculate $f(1, n)$, $f(2, n)$, $f(3, n)$ and find the pattern.

Answer: $f(4,1981) = 2^{2^{2^{\cdots^{2}}}} - 3$, a tower of 1984 2's less 3. In general, $f(4,n) = 2^{2^{2^{\cdots^{2}}}} - 3$, a tower of $(n+3)$ 2's less 3.

45. Hint: For avery real number t the triple $(a,b,c) = (6t, 3t, -2t)$ satisfy the condition $ab + bc + ca = 0$. What implication does it have on the equation?

Answer: $P(x) = \alpha x^4 + \beta x^2$ for any choice of real numbers α and β.

46. Hint: Start by taking $y = x$. Find out that $x + f(x) + xf(x)$ is a fixed point of $f(x)$ for each $x \in S$. How many fixed points can $f(x)$ has at most?

Answer: $f(x) = \dfrac{-x}{x+1}$.

47. Hint: Prove that $f(0) = 0$. Show that $f(f(y)) = y$ for all real y. Show thereafter that $f(x+y) = f(x) + f(y)$.

Answer: $f(x) = x$.

48. Hint: Start with $y = x$ and then change x to $f(x)$. Assume that $f\big(f(x)\big) > x$ and don't forget to use the fact that $f(x)$ is strictly decreasing.

49. Hint: Think of the numbers in base 2, i.e. let n_2 be the binary representation of n. Prove thereafter (using the induction) that the function $f(n_2)$ returns the number m_2 which has the same digits as n_2 but in the opposite order. The problem reduces then to finding the number of all integers ≤ 1988 with the symmetric binary representation. Find that the number of symmetrical binary numbers with k digits is $2^{\lfloor (n-1)/2 \rfloor}$. How many (binary) digits do we need in order to not exceed 1988?

Answer: 92.

50. Hint: Show that $f(x) = 0$ for all $x \geq 2$. How should $f(x)$ look like for $0 \leq x < 2$?

Answer: $f(x) = \begin{cases} \frac{2}{2-x} & \text{if } 0 \leq x < 2 \\ 0 & \text{if } x \geq 2 \end{cases}$

51. Hint: If $f(1) = k$, show that k divides $f(n)$ for all n. Show then that the function $g(n) = \dfrac{f(n)}{k}$ also satisfies teh given equation. Since we are looking for the smalest value of $f(1998)$ we may assume $f(1) = 1$.

Show then that if p is a prime number that $f(p)$ is a prime number as well and $f(f(p)) = p$. Show finaly that $f(n)$ can be defined arbitrary on primes as long as the conditions $f(p) = q$ (where p and q are prime) and $f(q) = p$ are satisfied.

Answer: $2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 120$.

52. Hint: Show that $f(xy) = f(x)f(y)$ and $f\big(f(x)\big) = \dfrac{1}{x}$ for all $x, y \in \mathbb{Q}^+$. For $x \in \mathbb{Q}^+$ construct $f(x)$ based on the prime faktorization of numerator and denumerator: $x = p_1^{n_1} p_2^{n_2} ... p_k^{n_k}$, where p_i are prime numbers and $n_i \in \mathbb{Z}$, since we have $f(x) = \big(f(p_1)\big)^{n_1}\big(f(p_2)\big)^{n_2} ... \big(f(p_k)\big)^{n_k}$ ($\star$). Thus it is enough to define a suitable function on the set of all prime numbers $\{p_1, p_2, p_3, ...\}$ and, of course, add $f(1) = 1$.

Answer: One possible construction is $f(p_j) = \begin{cases} p_{j+1} & \text{if } j \text{ is odd,} \\ \frac{1}{p_{j+1}} & \text{if } j \text{ is even} \end{cases}$, $f(1) = 1$ and extend it to whole $\mathbb{Q}^+$ using ($\star$).